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Heat kernel expansion coefficient: I. An extension

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Abstract. We argue that it is possible to obtain the asymptotic expansion coefficients for an operator with torsion from the expansion coefficients for an operator without torsion. This is possible when both operators act on the same set of eigenfunctions in the same way, and therefore have the same spectrum. We calculate in four dimensions $[a_2(-\Box+B^\kappa\nabla_\kappa+X)]$ using the Schwinger-DeWitt ansatz. Then following the argument above we obtain $[a_2(-\tilde{\Box}-\tilde{B}^\kappa\tilde{\nabla}_\kappa+X)]$ where tildes denote torsion. We compare our results with a modest direct calculation using 'toy torsion' (totally antisymmetric and covariantly constant) and obtain agreement. We discuss our results and review the literature. We find no other algorithms for $[a_2(-\tilde{\Box}+\tilde{B}^\kappa\tilde{\nabla}_\kappa+X)]$ consistent with the limiting case $[a_2(-\Box+B^\kappa\nabla_\kappa+X)]$. There are three appendices with useful coincidence limits and curvature tensor relations.

1. Introduction

This is the first of two papers contending with the $a_{1/2d}$ asymptotic expansion coefficient of the heat kernel for an associated non-negative elliptical operator on a d-dimensional manifold. This coefficient is of interest because it is related to the index theorem, the trace and axial anomalies and the logarithmic divergences at the first loop. With such special properties it is rightly called a magical coefficient (Christensen 1984). In this paper we show another magical property and use this to obtain relations for the $(d=4)$ a_2 coefficient of the operator $(-\tilde{\Box}\delta^i_j+[\tilde{B}^\kappa]^i_j\tilde{\nabla}_\kappa+X^i_j)$, where i, j, are spin, or group, indices and the tildes denote that torsion is present in the derivatives. This is the most general second-order operator with the leading symbol given by some power of the metric tensor.

In order to obtain our results, we make use of the observation that if two operators Δ_1 and Δ_2 act on eigenfunctions such that each operator has the same eigenvalue associated with the same eigenfunction, then these two operators have, by definition, the same spectrum. Therefore their asymptotic expansions must be equivalent. From this, we suggest that it is possible to obtain $[a_2(-\tilde{\Box}\delta^i_j+[\tilde{B}^\kappa]^i_j\tilde{\nabla}_\kappa+X^i_j)]$ from the torsion-free case $[a_2(-\Box\delta^i_j+[B^\kappa]^i_j\nabla_\kappa+X^i_j)]$. We calculate $[a_2(-\Box\delta^i_j+[B^\kappa]^i_j\nabla_\kappa+X^i_j)]$ directly from the Schwinger-DeWitt ansatz (which as far as we know has not been done before).

In § 2, we discuss notation and do necessary preliminary 'groundwork'. In § 3 we show how to find other second-order operators having equivalent spectra to the torsionful operator $(-\tilde{\Box}\delta^i_j+[\tilde{B}^\kappa]^i_j\tilde{\nabla}_\kappa+X^i_j)$. In § 4, we do the direct calculation, using the Schwinger-DeWitt ansatz, of $[a_2(-\Box\delta^i_j+[B^\kappa]^i_j\nabla_\kappa+X^i_j)]$. We quote results for the

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two cases where $[B^\kappa, B^\epsilon] \neq 0$, $[B^\kappa, X] \neq 0$, etc, and where these commutation relations are zero. Then, using these results, we find $[a_2(-\tilde{\Box}\delta^i_j + [\tilde{B}^\kappa]^i_j \tilde{\nabla}_\kappa + X^i_j)]$ for both arbitrary and totally antisymmetric torsion.

In § 5 we compare our results with a modest direct calculation of $[a_2(-\tilde{\Box}\delta^i_j + X^i_j)]$ for a Riemann flat manifold with 'toy torsion' which is totally antisymmetric and covariantly constant, and obtain agreement.

Finally, in § 6, we discuss our results, review the literature, and point to areas of future investigation. Among other things, we find that other results for $[a_2(-\tilde{\Box}\delta^i_j + [\tilde{B}^\kappa]^i_j \tilde{\nabla}_\kappa + X^i_j)]$ do *not*, in the limit of zero torsion, give $[a_2(-\Box\delta^i_j + [B^\kappa]^i_j \nabla_\kappa + X^i_j)]$.

2. Notation and preliminaries

The notation used here follows, as much as possible, that of Misner *et al* (1973) and Barth and Christensen (1983). Objects with a tilde denote torsion. Throughout, we assume a compact manifold without boundary and that given the metric $g_{\alpha\beta}$ the covariant derivative $\tilde{\nabla}_\kappa$ is metric compatible:

$$\tilde{\nabla}_\kappa g_{\alpha\beta} = 0 = g_{\alpha\beta \stackrel{:}{,} \kappa}. \tag{2.1}$$

Notice the shorthand notation here is a colon above a comma for covariant derivative with torsion. It is easily shown from (2.1) that:

$$\tilde{\Gamma}^\kappa{}_{\alpha\beta} = \{\Gamma^\kappa{}_{\alpha\beta} + \tfrac{1}{2}(T_{\alpha\beta}{}^\kappa - T^\kappa{}_{\alpha\beta} - T^\kappa{}_{\beta\alpha})\} \tag{2.2}$$

where $\Gamma^\kappa{}_{\alpha\beta}$ is the Christoffel symbol, or, as we often refer to it below, the symmetric connection, and $T_{\alpha\beta\kappa}$ is the torsion tensor with the properties

$$T_{\alpha\beta}{}^\kappa = T_{[\alpha\beta]}{}^\kappa = \{\tilde{\Gamma}^\kappa{}_{\alpha\beta} - \tilde{\Gamma}^\kappa{}_{\beta\alpha}\} \tag{2.3}$$

with brackets denoting antisymmetrisation:

$$T_{[\alpha\beta]}{}^\kappa = \tfrac{1}{2}(T_{\alpha\beta}{}^\kappa - T_{\beta\alpha}{}^\kappa). \tag{2.4}$$

Defining the contorsion tensor:

$$K_{\alpha\beta}{}^\kappa = \tfrac{1}{2}(T_{\alpha\beta}{}^\kappa - T^\kappa{}_{\alpha\beta} - T^\kappa{}_{\beta\alpha}) \tag{2.5}$$

equation (2.2) is now written as

$$\tilde{\Gamma}^\kappa{}_{\alpha\beta} = \{\Gamma^\kappa{}_{\alpha\beta} + K_{\alpha\beta}{}^\kappa\} \tag{2.6}$$

where the contorsion tensor has the symmetry

$$K_{\alpha\beta\kappa} = -K_{\kappa\beta\alpha}. \tag{2.7}$$

The Riemann curvature tensor is defined in the usual way:

$$\tilde{R}^\alpha{}_{\beta\kappa\epsilon} = \{\tilde{\Gamma}^\alpha{}_{\beta\epsilon,\kappa} - \tilde{\Gamma}^\alpha{}_{\beta\kappa,\epsilon} + \tilde{\Gamma}^\rho{}_{\beta\epsilon}\tilde{\Gamma}^\alpha{}_{\rho\kappa} - \tilde{\Gamma}^\rho{}_{\beta\kappa}\tilde{\Gamma}^\alpha{}_{\rho\epsilon}\} \tag{2.8}$$

and has the symmetries

$$\tilde{R}_{\alpha\beta\kappa\epsilon} = \tilde{R}_{[\alpha\beta][\kappa\epsilon]}. \tag{2.9}$$

However, because the connection $\tilde{\Gamma}$ is no longer symmetric $\tilde{R}_{\alpha\beta\kappa\varepsilon}$ is no longer symmetric under interchange of the index pairs $\alpha\beta$ and $\kappa\varepsilon$. The Ricci tensor and Riemann scalar are obtained in the usual way:

$$\tilde{R}_{\alpha\beta} = \tilde{R}_{\alpha\kappa\beta}{}^{\kappa} \qquad \tilde{R} = \tilde{R}_{\kappa}{}^{\kappa} = \tilde{R}_{\kappa\varepsilon}{}^{\kappa\varepsilon}. \tag{2.10}$$

The Ricci tensor is also no longer symmetric.

When torsion is present, the various identities are altered. The Bianchi identity becomes

$$\{\tilde{R}_{\alpha\beta\mu\nu;\kappa} + \tilde{R}_{\alpha\beta\nu\kappa;\mu} + \tilde{R}_{\alpha\beta\kappa\mu;\nu}\} = \{-T_{\mu\nu}{}^{\rho}\tilde{R}_{\alpha\beta\kappa\rho} - T_{\nu\kappa}{}^{\rho}\tilde{R}_{\alpha\beta\mu\rho} - T_{\kappa\mu}{}^{\rho}\tilde{R}_{\alpha\beta\nu\rho}\}. \tag{2.11}$$

The Ricci identity (an extra term arises when the covariant derivative is invariant under some other group transformation—this is discussed later) for some tensor $A_{\alpha\ldots\beta}$ is

$$[\tilde{\nabla}_{\kappa}, \tilde{\nabla}_{\varepsilon}]A_{\alpha\ldots\beta} = \{T_{\kappa\varepsilon}{}^{\rho}A_{\alpha\ldots\beta;\rho} + A_{\rho\ldots\beta}\tilde{R}_{\alpha}{}^{\rho}{}_{\kappa\varepsilon} + \ldots + A_{\kappa\ldots\rho}\tilde{R}_{\beta}{}^{\rho}{}_{\kappa\varepsilon}\}. \tag{2.12}$$

Additional identities and their contractions can be found in appendix 2. Naturally deleting the torsion terms, and taking off the tildes in these identities, one regains the identities for Riemannian manifolds.

Using the Christoffel symbol Γ one can also define the torsionless covariant derivative which from (2.1), (2.6) and (2.7) is easily shown to be also metric compatible. This torsionless covariant derivative is denoted with the usual semicolon. Using (2.6) it is possible to relate the two derivatives $\tilde{\nabla}_{\kappa}$ and ∇_{κ} to obtain the following relation where $A^{\alpha}{}_{\beta}$ is some arbitrary tensor:

$$A^{\alpha}{}_{\beta\tilde{;}\kappa} = \{A^{\alpha}{}_{\beta;\kappa} - K_{\beta\kappa}{}^{\varepsilon}A^{\kappa}{}_{\varepsilon} + K^{\alpha}{}_{\kappa\varepsilon}A^{\varepsilon}{}_{\beta}\}. \tag{2.13}$$

This is easily extended to tensors of any rank. Similarly, the torsionful and torsionless Riemann curvature tensors and their contractions can also be related using (2.6) and (2.8). The result is

$$R_{\alpha\beta\mu\nu} = \{\tilde{R}_{\alpha\beta\mu\nu} + K_{\beta\mu\alpha;\nu} - K_{\beta\nu\alpha;\mu} + K_{\beta\nu}{}^{\kappa}K_{\kappa\mu\alpha} - K_{\beta\mu}{}^{\kappa}K_{\kappa\nu\alpha}$$
$$+ K_{\mu\nu}{}^{\kappa}K_{\beta\kappa\alpha} - K_{\nu\mu}{}^{\kappa}K_{\beta\kappa\alpha}\} \tag{2.14a}$$
$$= \{\tilde{R}_{\alpha\beta\mu\nu} + M_{\alpha\beta\mu\nu}\}. \tag{2.14b}$$

Other expressions for the Ricci tensor and Riemann scalar as well as the various identities can all be found in appendix 2. It is also possible to write (2.14) in terms of the torsion tensor itself, rather than the contorsion tensor as is done here. This results in longer expressions which are given in Barth and Christensen (1983).

When acting on a spinor, the covariant derivatives, if they are to remain invariant under the associated spin group, must be altered by the introduction of a spin connection where here i and j are spin, or group indices, so that the Ricci identity is changed:

$$[\tilde{\nabla}_{\kappa}, \tilde{\nabla}_{\varepsilon}][A_{\alpha\ldots\beta}]^{i} = \{[Y_{\kappa\varepsilon}]^{i}_{j}[A_{\alpha\ldots\beta}]^{j} + T_{\kappa\varepsilon}{}^{\rho}[A_{\alpha\ldots\beta}]^{i}{}_{;\rho}$$
$$+ [A_{\rho\ldots\beta}]^{i}\tilde{R}_{\alpha}{}^{\rho}{}_{\kappa\varepsilon}\ldots[A_{\alpha\ldots\rho}]^{i}\tilde{R}_{\beta}{}^{\rho}{}_{\kappa\varepsilon}\} \tag{2.15}$$

where $[Y_{\kappa\varepsilon}]^{i}_{j}$ is the group curvature constructed in the usual way. See also DeWitt (1965) and Goldthorpe (1980).

A final notational point: throughout, the coincidence limit of an expression, $\lim_{x'\to x}$, will always be given by brackets [ ] of the expression as in Christensen (1976) and Synge (1960).

3. The Schwinger–DeWitt proper time method

The Schwinger–DeWitt proper time method is, by now, well known (Barvinsky and Vilkovisky 1985, DeWitt 1965, 1975). However, some important relations for elliptical operators and heat kernels are the following. Given the elliptical operator Δ on a compact manifold of dimension d, with spectral decomposition $\{\lambda_i, \phi_i\}$ of eigenvalues and eigenvectors forming a complete orthonormal system such that

$$\Delta\phi_i = \lambda_i\phi_i \tag{3.1}$$

then there exists a heat kernel

$$K(x, x', s) = e^{-i\Delta s} \tag{3.2}$$

solving the heat equation:

$$\left(\frac{\partial}{\partial(is)} + \Delta\right)K(x, x', s) = 0 \tag{3.3}$$

which has the asymptotic expansion for $s \to 0^+$:

$$\mathrm{Tr}\, K(x, x', s) \sim \sum_{l=0}^{\infty} a_l(x, x')s^{(2l-d)/p} \tag{3.4}$$

where $p = \mathrm{O}(\Delta)$. When $\Delta = (-\Box\delta^i_j + [B^\kappa]^i_j\nabla_\kappa + X^i_j)$ the expansion coefficients (3.4) and the heat equation (3.3) can be solved using the Schwinger–DeWitt ansatz:

$$K(x, x', s) = (4\pi is)^{(-1/2)d}\, e^{-(1/2is)\sigma(x,x')}\Omega(x, x', s) \tag{3.5}$$

where

$$\Omega(x, x', s) = \sum_{l=0}^{\infty} a_l(x, x')(is)^l \tag{3.6}$$

and $\sigma(x, x')$ is the geodetic interval (DeWitt 1965, Synge 1960). Often the Van Vleck–Morette determinant factor appears in (3.5), but this is unnecessary as it can be absorbed into the coefficients (3.6) (I thank S M Christensen for pointing this out to me in a private conversation).

In d dimensions, it is possible to show that the $a_{(1/2)d}$ coefficient of (3.4) has the property:

$$\int_M g^{1/2}\, d^4x\, a_{(1/2)d} \equiv A_{(1/2)d} = (n + m) \tag{3.7}$$

where n is the number of zero eigenvalues and m is the number of non-zero eigenvalues satisfying (3.1) (Christensen and Duff 1979, Hawking 1977). In addition to (3.7) other interesting relationships for this coefficient are the trace anomaly

$$T^\kappa_\kappa = \tfrac{1}{2}(-1)^{2A+2B}\{a_{1/2}(A, B) + a_{1/2}(B, A)\} \tag{3.8}$$

and the axial anomaly

$$\nabla^\kappa J^5_\kappa = \{a_{(1/2)d}(A, B) - a_{(1/2)d}(B, A)\} \tag{3.9}$$

where A and B refer to the (A, B) representation of the Lorentz group (see Christensen and Duff (1979) for details). It is also related to the index theorem. Finally, the logarithmic diverges at the first loop level due to closed loops in the gravitational background field are

$$\Delta L^{(1)} = (n-d)^{-1}\tfrac{1}{2}(-1)^{2A+2B}\{a_{(1/2)d}(A, B) - a_{(1/2)d}(B, A)\} \tag{3.10}$$

(see e.g. Christensen and Duff (1979)). In all of the above it is also possible to generalise to operators $\Delta_{ij\ldots k}$ where i, j and k stand for some combination of indices. However, as the case without these extra indices can be considered without loss of generality we will not consider their unnecessary complication.

The most general second-order differential operator with the leading symbol given by the metric tensor is the torsion operator

$$\tilde{\Delta} = (-\tilde{\Box} + \tilde{B}^{\kappa}\tilde{\nabla}_{\kappa} + X) \tag{3.11}$$

where the tensor $\tilde{B}^{\kappa}$ is a first-order object (e.g. $\tilde{B}^{\kappa} = T^{\kappa}{}_{\varepsilon}{}^{\varepsilon}$), and X is some second-order object. In what follows the X piece is unimportant, and for now we discard it. When acting on a scalar field ϕ this operator (3.11) gives, using (2.6):

$$\tilde{\Delta}\phi = \{-g^{\alpha\beta}\phi_{,\alpha\beta} + (g^{\alpha\beta}\Gamma^{\kappa}{}_{\alpha\beta} + g^{\alpha\beta}K_{\alpha\beta}{}^{\kappa} + \tilde{B}^{\kappa})\phi_{,\kappa}\}. \tag{3.12}$$

Now we define a new connection $\acute{\Gamma}$ which has the properties:

$$g^{\alpha\beta}\acute{\Gamma}^{\kappa}{}_{\alpha\beta} = (g^{\alpha\beta}\Gamma^{\kappa}{}_{\alpha\beta} + g^{\alpha\beta}K_{\alpha\beta}{}^{\kappa} + \tilde{B}^{\kappa}) \tag{3.13a}$$

$$= g^{\alpha\beta}(\Gamma^{\kappa}{}_{\alpha\beta} + \acute{K}_{\alpha\beta}{}^{\kappa}) \tag{3.13b}$$

where the new tensor $\acute{K}_{\alpha\beta}{}^{\kappa}$ will turn out to be a new contorsion tensor. The other necessary property is that the associated covariant derivative $\acute{\nabla}$ is metric compatible: $\acute{\nabla}_{\kappa}g_{\alpha\beta} = 0$. This holds if and only if the tensor $\acute{K}_{\alpha\beta\kappa}$ has the antisymmetry property: $\acute{K}_{\alpha\beta\kappa} = -\acute{K}_{\kappa\beta\alpha}$. Notice that this is exactly what is true of the usual contorsion tensor of (2.7). To determine $\acute{K}_{\alpha\beta\kappa}$ completely, we use this antisymmetry property together with (3.13) to obtain

$$\acute{K}_{\alpha\beta\kappa} = \{K_{\alpha\beta\kappa} + \tfrac{1}{3}g_{\alpha\beta}\tilde{B}_{\kappa} - \tfrac{1}{3}g_{\kappa\beta}\tilde{B}_{\alpha} + A_{\alpha\beta\kappa}\} \tag{3.14}$$

where the tensor $A_{\alpha\beta\kappa}$ has the properties

$$g^{\alpha\beta}A_{\alpha\beta\kappa} = 0 \qquad A_{\alpha\beta\kappa} = -A_{\kappa\beta\alpha}. \tag{3.15}$$

To determine $A_{\alpha\beta\kappa}$ consider the special case where $\acute{\Gamma}^{\kappa}{}_{\alpha\beta} = \Gamma^{\kappa}{}_{\alpha\beta}$ which happens when $\tilde{B}^{\kappa} = -K^{\varepsilon}{}_{\varepsilon}{}^{\kappa}$. Then from (3.13) one has

$$\acute{K}_{\alpha\beta\kappa} = 0 = \{K_{\alpha\beta\kappa} - \tfrac{1}{3}g_{\alpha\beta}K_{\varepsilon}{}^{\varepsilon}{}_{\kappa} + \tfrac{1}{3}g_{\kappa\beta}K_{\varepsilon}{}^{\varepsilon}{}_{\alpha} + A_{\alpha\beta\kappa}\} \tag{3.16}$$

which gives

$$A_{\alpha\beta\kappa} = \{-K_{\alpha\beta\kappa} + \tfrac{1}{3}g_{\alpha\beta}K_{\varepsilon}{}^{\varepsilon}{}_{\kappa} - \tfrac{1}{3}g_{\kappa\beta}K_{\varepsilon}{}^{\varepsilon}{}_{\alpha}\} \tag{3.17}$$

which determines $A_{\alpha\beta\kappa}$. Notice that $A_{\alpha\beta\kappa}$ satisfies all the necessary properties (3.15) as it should. The full form for $\acute{K}_{\alpha\beta\kappa}$ then is simply

$$\acute{K}_{\alpha\beta\kappa} = \{\tfrac{1}{3}g_{\alpha\beta}(\tilde{B}_{\kappa} + K_{\varepsilon}{}^{\varepsilon}{}_{\kappa}) - \tfrac{1}{3}g_{\kappa\beta}(\tilde{B}_{\alpha} + K_{\varepsilon}{}^{\varepsilon}{}_{\alpha})\}. \tag{3.18}$$

The primed covariant derivative is now completely determined (and metric compatible) so that (3.11) becomes (with discarded X term)

$$(-\tilde{\Box} + \tilde{B}^{\kappa}\tilde{\nabla}_{\kappa})\phi = -\acute{\Box}\phi. \tag{3.19}$$

Defining a new tensor B^κ as follows:

$$B^\kappa \equiv (\tilde{B}^\kappa + K^{\epsilon}{}_{\epsilon}{}^{\kappa}) \tag{3.20}$$

we also have, from (3.13), (3.18) and (3.19), the relation

$$(-\tilde{\Box} + \tilde{B}^\kappa \tilde{\nabla}_\kappa)\phi = (-\Box + B^\kappa \nabla_\kappa)\phi = -\acute{\Box}\phi. \tag{3.21}$$

Following the same procedure it is possible to build a double primed derivative such that

$$\acute{\acute{\Gamma}}{}^{\kappa}{}_{\alpha\beta} = (\tilde{\Gamma}^{\kappa}{}_{\alpha\beta} + \acute{\acute{K}}_{\alpha\beta}{}^{\kappa}). \tag{3.22}$$

The relationships between the various contorsion and torsion tensors are then simply

$$\acute{\acute{K}}_{\alpha\beta\kappa} = \{\acute{K}_{\alpha\beta\kappa} - K_{\alpha\beta\kappa}\} \tag{3.23a}$$

$$\acute{\acute{T}}_{\alpha\beta\kappa} = \{\acute{T}_{\alpha\beta\kappa} - T_{\alpha\beta\kappa}\} \tag{3.23b}$$

$$\acute{T}_{\alpha\beta\kappa} = \acute{K}_{\alpha\kappa\beta}. \tag{3.23c}$$

Equation (3.21) can now be extended to give

$$(-\tilde{\Box} + \tilde{B}^\kappa \tilde{\nabla}_\kappa)\phi = (-\Box + B^\kappa \nabla_\kappa)\phi = -\acute{\Box}\phi = -\acute{\acute{\Box}}\phi. \tag{3.24}$$

Although from (3.24) it is clear that the prime and double prime operators act on scalars in the same way, their contorsion tensors are different and the 'decomposition' of their connections (3.13*b*) and (3.22) respectively are different. However both of these new derivatives $\acute{\nabla}$ and $\acute{\acute{\nabla}}$ have the very same properties of the usual torsionful derivatives. That means all of the identities and relations given in § 2 for torsion are satisfied by the prime and double prime derivatives as well. Similarly this is also true for the construction of the Riemann curvature tensor and its contractions. Some relations are given explicitly in appendix 2.

Due to the relationships (3.24) these operators will have equivalent spectral decompositions solving the eigenvalue equation (3.1). Thus the number of zero and non-zero modes will be the same for these operators, and therefore the $a_{(1/2)d}$ coefficients associated with these operators (3.24) will be equivalent, though simply written in terms of different objects. For example, the a_2 coefficient for the torsion operator (leftmost) in (3.24) will be in terms of the fourth-order invariants given in Christensen (1980) whereas for the torsionless operator of (3.24) it will be in terms of those torsionless objects built from the Riemann tensor, its contractions and derivatives. It is then possible to convert one a_2 coefficient into the other by simply rewriting the various torsionless objects in terms of those with torsion, using relation (3.19) and those in appendix 2.

What is to be gained by all of this? The direct calculation of the a_2 coefficient (we are assuming four dimensions for now) using (3.5) and (3.6) is very hard to do in the torsion case. This is largely due to the significant number of fourth-order invariants possible when torsion is introduced. In the torsionless case, there are relatively far fewer fourth-order invariants. This, together with the fact that the various identities (such as the Ricci) have a much simpler form, results in an easier calculation which can be carried out faster and with better accuracy, and can later be converted into an expression for the torsion case. We shall use this property in the next section to obtain results for the operator (3.11).

A final point is that, although we have focused on the $a_{(1/2)d}$ coefficient with its special properties, because the spectra of the operators (3.24) are all the same this suggests that their entire asymptotic expansions are equivalent. One expects therefore a similar procedure of rewriting torsionless asymptotic coefficients in terms of torsionful invariants to obtain the torsionful asymptotic coefficients to also hold. In the next section (which takes place in four dimensions) this will be borne out for the a_1 coefficient (as well as the a_2 coefficient which is to be our main focus) and also the a_0 coefficient. Thus, at least for these first coefficients, it seems our expectations for the asymptotic expansion coefficients other than the $a_{(1/2)d}$ coefficient are true. Having said this we move on to the next section where we shall focus exclusively on the $a_{(1/2)d}$ coefficient.

4. The calculation

We now specialise to four dimensions and calculate the a_2 coefficient for the operator given in (3.11). One could consider calculating this directly. Substituting the Schwinger-DeWitt ansatz (3.5) into the heat equation (3.3) one obtains recursion relations and an indicial equation allowing one to solve for, in principle, all of the asymptotic expansion coefficients (3.4). This 'straightforward' method has been used by Goldthorpe (1980) and Nieh and Yan (1982). As we have mentioned before, it has the disadvantage that when torsion is involved this direct calculation strains normal humans to their limits. For this reason another method was suggested in § 3. This method we now follow through here.

Due to the relation (3.24) we calculate a_2 for the opeiator

$$(-\Box + B^{\kappa}\nabla_{\kappa} + X) \tag{4.1}$$

instead of for the operator (3.11). Operator (4.1) has no torsion and so it is much easier to find a_2 for this operator. We do this in a 'straightforward' way, substituting (3.5) into (3.3) obtaining the indicial equation

$$\{-\tfrac{1}{2}na_0 + \sigma_{;}{}^{\kappa}a_{0;\kappa} + \tfrac{1}{2}\sigma_{;\kappa}{}^{\kappa}a_0 - \tfrac{1}{2}B^{\kappa}\sigma_{;\kappa}a_0\} = 0 \tag{4.2}$$

and recursion relations

$$\{-\tfrac{1}{2}na_{l+1} + (l+1)a_{l+1} + \sigma_{;}{}^{\kappa}a_{l+1;\kappa} + \tfrac{1}{2}\sigma_{;\kappa}{}^{\kappa}a_{l+1} - a_{l;\kappa}{}^{\kappa} - \tfrac{1}{2}B^{\kappa}\sigma_{;\kappa}a_{l+1} + B^{\kappa}a_{l;\kappa} + Xa_l\} = 0. \tag{4.3}$$

Note that the group indices i and j are left implicit in equations (4.2) and (4.3). This can be done without loss of generality. Taking derivatives of (4.2) and (4.3), and using the coincidence limits of $\sigma(x, x')$ and the geodetic interval given in appendix 1, one easily obtains the necessary expressions:

$$[a_{0;\kappa}] = \tfrac{1}{2}B_{\kappa} \tag{4.4}$$

$$[a_{0;\kappa\varepsilon}] = -\tfrac{1}{2}\{Y_{\kappa\varepsilon} - \tfrac{2}{3}R_{\kappa\varepsilon} - \tfrac{1}{2}B_{\kappa;\varepsilon} - \tfrac{1}{2}B_{\varepsilon;\kappa} - \tfrac{1}{4}B_{\kappa}B_{\varepsilon} - \tfrac{1}{4}B_{\varepsilon}B_{\kappa}\} \tag{4.5}$$

$$\begin{aligned}[a_{0;\kappa}{}^{\kappa}{}_{\varepsilon}] = -\tfrac{1}{3}\{ & Y_{\kappa\varepsilon}B^{\kappa} + \tfrac{1}{2}B^{\kappa}Y_{\kappa\varepsilon} + Y_{\kappa\varepsilon;}{}^{\kappa} - \tfrac{1}{2}R_{;\varepsilon} - \tfrac{1}{4}RB_{\varepsilon} - \tfrac{1}{4}B_{\varepsilon}B^{\kappa}{}_{;\kappa} \\ & - \tfrac{1}{8}B_{\varepsilon}B^{\kappa}B_{\kappa} - \tfrac{1}{4}B^{\kappa}B_{\kappa;\varepsilon} - \tfrac{1}{4}B^{\kappa}B_{\varepsilon;\kappa} - \tfrac{1}{8}B^{\kappa}B_{\kappa}B_{\varepsilon} - \tfrac{1}{8}B^{\kappa}B_{\varepsilon}B_{\kappa} \\ & - \tfrac{1}{2}B_{\varepsilon;}{}^{\kappa}{}_{\kappa} - B_{\kappa;}{}^{\kappa}{}_{\varepsilon} - \tfrac{1}{2}B^{\kappa}{}_{;\kappa}B_{\varepsilon} - \tfrac{1}{2}B_{\varepsilon;\kappa}B^{\kappa} - \tfrac{1}{2}B^{\kappa}{}_{;\varepsilon}B^{\kappa}\}\end{aligned} \tag{4.6}$$

$$
\begin{aligned}
[a_{0;\kappa}{}^{\kappa}{}_{\varepsilon}{}^{\varepsilon}] = -\tfrac{1}{4}\{ & -2Y_{\kappa\varepsilon}Y^{\kappa\varepsilon} - 2Y_{\kappa\varepsilon}B^{\kappa}{}_{;}{}^{\varepsilon} - Y_{\kappa\varepsilon}B^{\kappa}B^{\varepsilon} + B^{\kappa}Y_{\kappa\varepsilon}B^{\varepsilon} + B^{\kappa}Y_{\kappa\varepsilon;}{}^{\varepsilon} + \tfrac{1}{2}[\sigma_{;\kappa}{}^{\kappa}{}_{\varepsilon}{}^{\varepsilon}{}_{\alpha}{}^{\alpha}] \\
& - R_{;\kappa}B^{\kappa} - \tfrac{1}{9}R^2 - \tfrac{2}{3}RB_{\kappa;}{}^{\kappa} - \tfrac{1}{6}RB_{\kappa}B^{\kappa} - 2B^{\kappa}{}_{;\kappa}{}^{\varepsilon}{}_{\varepsilon} - \tfrac{4}{3}B^{\kappa}{}_{;}{}^{\varepsilon}R_{\kappa\varepsilon} \\
& - B^{\kappa}{}_{;\varepsilon}{}^{\varepsilon}B_{\kappa} - 2B^{\kappa}{}_{;\kappa}{}^{\varepsilon}B_{\varepsilon} - \tfrac{4}{3}B^{\kappa}B^{\varepsilon}R_{\kappa\varepsilon} - B^{\kappa}{}_{;\kappa}B^{\varepsilon}{}_{;\varepsilon} - \tfrac{1}{2}B^{\kappa}{}_{;\kappa}B^{\varepsilon}B_{\varepsilon} \\
& - 2B^{\kappa}[a_{0;\varepsilon}{}^{\varepsilon}{}_{\kappa}] - B^{\kappa}{}_{;}{}^{\varepsilon}B_{\kappa;\varepsilon} - B^{\kappa}{}_{;}{}^{\varepsilon}B_{\varepsilon;\kappa} \\
& - \tfrac{1}{2}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\kappa}B_{\varepsilon} - \tfrac{1}{2}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\varepsilon}B_{\kappa}\}
\end{aligned} \tag{4.7}
$$

$$
[a_1] = \{[a_{0;\kappa}{}^{\kappa}] - B^{\kappa}[a_{0;\kappa}] - X\} \tag{4.8}
$$

$$
\begin{aligned}
12[a_2] = \{ & 2[a_{0;\kappa}{}^{\kappa}{}_{\varepsilon}{}^{\varepsilon}] - 2B^{\kappa}[a_{0;\kappa}{}^{\varepsilon}{}_{\varepsilon}] - 2B^{\kappa}[a_{0;\varepsilon}{}^{\varepsilon}{}_{\kappa}] - [\sigma_{;\kappa}{}^{\kappa}{}_{\varepsilon}{}^{\varepsilon}][a_1] + 2B^{\kappa}{}_{;\kappa}[a_1] \\
& - B^{\kappa}B_{\kappa}[a_1] - 6X[a_1] - 4B^{\kappa}{}_{;}{}^{\varepsilon}[a_{0;\kappa\varepsilon}] + 2B^{\kappa}B^{\varepsilon}[a_{0;\varepsilon\kappa}] - 2X[a_{0;\kappa}{}^{\kappa}] \\
& - B^{\kappa}{}_{;\varepsilon}{}^{\varepsilon}B_{\kappa} - 2X_{;\kappa}{}^{\kappa} - 2X_{;}{}^{\kappa}B_{\kappa} + B^{\kappa}B^{\varepsilon}{}_{;\kappa}B_{\varepsilon} + 2B^{\kappa}X_{;\kappa} + B^{\kappa}XB_{\kappa}\}.
\end{aligned} \tag{4.9}
$$

It is important to remember that due to the existence of the group indices, the generalised Ricci identity (2.15) must be used. Then using (4.4) through (4.9) one obtains

$$
[a_1] = \{\tfrac{1}{6}R + \tfrac{1}{2}B_{\kappa;}{}^{\kappa} - \tfrac{1}{4}B^{\kappa}B_{\kappa} - X\} \tag{4.10}
$$

as well as

$$
\begin{aligned}
& 180[a_2(-\Box + B^{\kappa}\nabla_{\kappa} + X)] \\
& \quad = \{R_{\alpha\beta\kappa\varepsilon}R^{\alpha\beta\kappa\varepsilon} - R_{\alpha\beta}R^{\alpha\beta} + \tfrac{5}{2}R^2 + 6R_{;\kappa}{}^{\kappa} + 90X^2 - 30XR - 30X_{;\kappa}{}^{\kappa} + 15Y_{\kappa\varepsilon}Y^{\kappa\varepsilon} \\
& \qquad - \tfrac{15}{2}B^{\kappa}B^{\varepsilon}Y_{\kappa\varepsilon} + \tfrac{15}{2}Y_{\kappa\varepsilon}B^{\kappa}B^{\varepsilon} + \tfrac{15}{2}B^{\kappa}Y_{\kappa\varepsilon}B^{\varepsilon} + 30B^{\kappa}{}_{;}{}^{\varepsilon}Y_{\kappa\varepsilon} + 15Y_{\kappa\varepsilon}B^{\kappa}{}_{;}{}^{\varepsilon} \\
& \qquad + \tfrac{15}{2}B^{\kappa}Y_{\kappa\varepsilon;}{}^{\varepsilon} + 15RB_{\kappa;}{}^{\kappa} - \tfrac{15}{2}RB^{\kappa}B_{\kappa} + 15B^{\kappa}B_{\kappa}X + 15XB^{\kappa}B_{\kappa} \\
& \qquad + 15B^{\kappa}XB_{\kappa} - 30B^{\kappa}{}_{;\kappa}X - 60XB^{\kappa}{}_{;\kappa} + 30B^{\kappa}X_{;\kappa} - 30X_{;\kappa}B^{\kappa} \\
& \qquad + 15B^{\kappa}{}_{;\kappa}{}^{\varepsilon}{}_{\varepsilon} - \tfrac{15}{2}B^{\kappa}{}_{;\varepsilon}{}^{\varepsilon}B_{\kappa} - \tfrac{15}{2}B^{\kappa}B_{\kappa;}{}^{\varepsilon}{}_{\varepsilon} + 15B^{\kappa}{}_{;\kappa\varepsilon}B^{\varepsilon} - 15B^{\varepsilon}B^{\kappa}{}_{;\kappa\varepsilon} + \tfrac{45}{2}B^{\kappa}{}_{;\kappa}B^{\varepsilon}{}_{;\varepsilon} \\
& \qquad - \tfrac{15}{2}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\kappa;\varepsilon} - \tfrac{15}{2}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\varepsilon;\kappa} - \tfrac{15}{4}B^{\kappa}{}_{;\kappa}B^{\varepsilon}B_{\varepsilon} - \tfrac{15}{4}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\kappa}B_{\varepsilon} - \tfrac{15}{4}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\varepsilon}B_{\kappa} \\
& \qquad + \tfrac{15}{4}B^{\kappa}B^{\varepsilon}B_{\kappa;\varepsilon} + \tfrac{15}{4}B^{\kappa}B^{\varepsilon}B_{\varepsilon;\kappa} - \tfrac{15}{2}B^{\kappa}B_{\kappa;\varepsilon}B^{\varepsilon} + \tfrac{15}{2}B^{\kappa}B_{\varepsilon;\kappa}B^{\varepsilon} - \tfrac{45}{4}B^{\varepsilon}B_{\varepsilon}B^{\kappa}{}_{;\kappa} \\
& \qquad - \tfrac{15}{2}B^{\varepsilon}B^{\kappa}{}_{;\kappa}B_{\varepsilon} + \tfrac{15}{8}B^{\kappa}B_{\kappa}B^{\varepsilon}B_{\varepsilon} + \tfrac{15}{8}B^{\kappa}B^{\varepsilon}B^{\kappa}B_{\varepsilon} + \tfrac{15}{8}B^{\kappa}B^{\varepsilon}B_{\varepsilon}B_{\kappa}\}.
\end{aligned} \tag{4.11}
$$

Notice that depending on how the tensors $[B^{\kappa}]^i_j$ and X^i_j are defined, they may or may not commute with themselves, or each other. The expressions (4.10) and (4.11) are the general expressions which assume that all such commutations are *not* zero. If $[B^{\kappa}, B^{\varepsilon}]_- = 0$, $[B^{\kappa}, X] = 0$, $[B_{\kappa}, Y_{\alpha\beta}]_- = 0$, $[X, Y_{\alpha\beta}]_- = 0$ then (3.15) becomes

$$
\begin{aligned}
180[a_2] = \{ & R_{\alpha\beta\kappa\varepsilon}R^{\alpha\beta\kappa\varepsilon} - R_{\alpha\beta}R^{\alpha\beta} + \tfrac{5}{2}R^2 + 6R_{;\kappa}{}^{\kappa} + 90X^2 - 30XR - 30X_{;\kappa}{}^{\kappa} + 15Y_{\kappa\varepsilon}Y^{\kappa\varepsilon} \\
& + 45B^{\kappa}{}_{;}{}^{\varepsilon}Y_{\kappa\varepsilon} + \tfrac{15}{2}B^{\kappa}Y_{\kappa\varepsilon;}{}^{\kappa} + 15RB_{\kappa;}{}^{\kappa} - \tfrac{15}{2}RB^{\kappa}B_{\kappa} + 15B^{\kappa}{}_{;\kappa}{}^{\varepsilon}{}_{\varepsilon} \\
& - 15B^{\kappa}B_{\kappa;}{}^{\varepsilon}{}_{\varepsilon} + \tfrac{45}{2}B^{\kappa}{}_{;\kappa}B^{\varepsilon}{}_{;\varepsilon} - \tfrac{15}{2}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\kappa;\varepsilon} - \tfrac{15}{2}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\varepsilon;\kappa} - \tfrac{45}{2}B^{\kappa}B_{\kappa}B^{\varepsilon}{}_{;\varepsilon} \\
& - 90B^{\kappa}{}_{;\kappa}X + 45B^{\kappa}B_{\kappa}X + \tfrac{45}{8}B^{\kappa}B_{\kappa}B^{\varepsilon}B_{\varepsilon}\}.
\end{aligned} \tag{4.12}
$$

Using the following relations:

$$
B_{\kappa;\varepsilon} = \{B_{\kappa|\varepsilon} + K_{\kappa\varepsilon}{}^{\rho}B_{\rho}\} \tag{4.13a}
$$

$$
Y_{\kappa\varepsilon;}{}^{\varepsilon} = \{Y_{\kappa\varepsilon|}{}^{\varepsilon} + K_{\kappa}{}^{\varepsilon\rho}Y_{\rho\varepsilon} + K^{\varepsilon}{}_{\varepsilon}{}^{\rho}Y_{\kappa\rho}\} \tag{4.13b}
$$

$$
B^{\kappa}{}_{\kappa;\varepsilon} = \{B^{\kappa}{}_{|\kappa\varepsilon} + K^{\alpha}{}_{\alpha}{}^{\kappa}{}_{|\varepsilon}B_{\kappa} + K^{\alpha}{}_{\alpha}{}^{\kappa}B_{\kappa|\varepsilon}\} \tag{4.13c}
$$

$$B_{\kappa;}{}^{\varepsilon}{}_{\varepsilon} = \{B_{\kappa;}{}^{\varepsilon}{}_{\varepsilon} + K_{\kappa}{}^{\varepsilon\rho}{}_{;\varepsilon}B_{\rho} + 2K_{\kappa}{}^{\varepsilon\rho}B_{\rho;\varepsilon} + K_{\kappa}{}^{\varepsilon\rho}K_{\rho\varepsilon}{}^{\lambda}B_{\lambda} + K^{\varepsilon}{}_{\varepsilon}{}^{\rho}K_{\kappa\rho}{}^{\lambda}B_{\lambda} + K^{\varepsilon}{}_{\varepsilon}{}^{\rho}B_{\kappa;\rho}\} \tag{4.13d}$$

$$\begin{aligned} B^{\kappa}{}_{;\kappa}{}^{\varepsilon}{}_{\varepsilon} = \{ & B^{\kappa}{}_{;\kappa}{}^{\varepsilon}{}_{\varepsilon} + K^{\alpha}{}_{\alpha}{}^{\kappa}{}_{;\varepsilon}{}^{\varepsilon}B_{\kappa} + 2K^{\alpha}{}_{\alpha}{}^{\kappa}{}_{;}{}^{\varepsilon}B_{\kappa;\varepsilon} + K^{\alpha}{}_{\alpha}{}^{\kappa}B_{\kappa;}{}^{\varepsilon}{}_{\varepsilon} \\ & + K^{\varepsilon}{}_{\varepsilon}{}^{\kappa}B^{\alpha}{}_{;\alpha\kappa} + K^{\varepsilon}{}_{\varepsilon}{}^{\kappa}K^{\alpha}{}_{\alpha}{}^{\rho}{}_{;\kappa}B_{\rho} + K^{\varepsilon}{}_{\varepsilon}{}^{\kappa}K^{\alpha}{}_{\alpha}{}^{\rho}B_{\kappa;\rho}\} \end{aligned} \tag{4.13e}$$

along with appendix 2, it is possible to convert (4.11) to obtain $[a_2(-\tilde{\Box} + \tilde{B}^{\kappa}\tilde{\nabla}_{\kappa} + X)]$. The equations (4.13) allow one to rewrite all of the B-dependent terms in (4.11). The relations in appendix 2 such as $R_{\alpha\beta\kappa\varepsilon} = (\tilde{R}_{\alpha\beta\kappa\varepsilon} + M_{\alpha\beta\kappa\varepsilon})$ allow one to rewrite all of the terms dependent on the Riemann tensor and its contractions in terms of the torsionful curvature tensor and the contorsion ($M_{\alpha\beta\kappa\varepsilon}$ depends on the contorsion as given in (A2.7)). When this is done, the result is

$$\begin{aligned}
&180[a_2(-\tilde{\Box} + \tilde{B}^{\kappa}\tilde{\nabla}_{\kappa} + X)] \\
&\quad = \{|\tilde{R}_{\alpha\beta\gamma\delta}|^2 - |\tilde{R}_{\alpha\beta}|^2 + \tfrac{5}{2}\tilde{R}^2 + 6\tilde{R}_{;\kappa}{}^{\kappa} + 90X^2 - 30X\tilde{R} - 30X_{;\kappa}{}^{\kappa} + 15Y_{\kappa\varepsilon}Y^{\kappa\varepsilon} \\
&\quad + 2\tilde{R}^{\alpha\beta\gamma\delta}M_{\alpha\beta\gamma\delta} - 2\tilde{R}^{\alpha\beta}M_{\alpha\beta} + 5\tilde{R}M + |M_{\alpha\beta\gamma\delta}|^2 - |M_{\alpha\beta}|^2 + \tfrac{5}{2}M^2 + 6M_{;\kappa}{}^{\kappa} \\
&\quad + 6K^{\varepsilon}{}_{\varepsilon}{}^{\kappa}\tilde{R}_{;\kappa} + 6K^{\varepsilon}{}_{\varepsilon}{}^{\kappa}M_{;\kappa} - 30XM - 30K^{\varepsilon}{}_{\varepsilon}{}^{\kappa}X_{;\kappa} + \tfrac{15}{2}B^{\kappa}B^{\varepsilon}Y_{\kappa\varepsilon} \\
&\quad + \tfrac{15}{2}Y_{\kappa\varepsilon}B^{\kappa}B^{\varepsilon} + \tfrac{15}{2}B^{\kappa}Y_{\kappa\varepsilon}B^{\varepsilon} + 30B^{\kappa}{}_{;}{}^{\varepsilon}Y_{\kappa\varepsilon} + 30K^{\kappa\varepsilon\rho}B_{\rho}Y_{\kappa\varepsilon} + 15Y_{\kappa\varepsilon}B^{\kappa}{}_{;}{}^{\varepsilon} \\
&\quad + 15Y_{\kappa\varepsilon}K^{\kappa\varepsilon\rho}B_{\rho} + \tfrac{15}{2}B^{\kappa}Y_{\kappa\varepsilon;}{}^{\varepsilon} + \tfrac{15}{2}B^{\kappa}K_{\kappa}{}^{\varepsilon\rho}Y_{\rho\varepsilon} + \tfrac{15}{2}B^{\kappa}K^{\varepsilon}{}_{\varepsilon}{}^{\rho}Y_{\kappa\rho} + 15\tilde{R}B^{\kappa}{}_{;\kappa} \\
&\quad + 15\tilde{R}K^{\varepsilon}{}_{\varepsilon}{}^{\kappa}B_{\kappa} + 15MB^{\kappa}{}_{;\kappa} + 15MK^{\varepsilon}{}_{\varepsilon}{}^{\kappa}B_{\kappa} - \tfrac{15}{2}\tilde{R}B^{\kappa}B_{\kappa} - \tfrac{15}{2}MB^{\kappa}B_{\kappa} \\
&\quad + 15B^{\kappa}B_{\kappa}X + 15XB^{\kappa}B_{\kappa} + 15B^{\kappa}XB_{\kappa} - 30B^{\kappa}{}_{;\kappa}X - 30K^{\varepsilon}{}_{\varepsilon}{}^{\kappa}B_{\kappa}X - 60XB^{\kappa}{}_{;\kappa} \\
&\quad - 60K^{\varepsilon}{}_{\varepsilon}{}^{\kappa}XB_{\kappa} + 30B^{\kappa}X_{;\kappa} - 30X_{;\kappa}B^{\kappa} + 15B^{\kappa}{}_{;\kappa}{}^{\varepsilon}{}_{\varepsilon} + 15K^{\alpha}{}_{\alpha}{}^{\kappa}{}_{;\varepsilon}{}^{\varepsilon}B_{\kappa} \\
&\quad + 30K^{\alpha}{}_{\alpha}{}^{\kappa}{}_{;}{}^{\varepsilon}B_{\kappa;\varepsilon} + 15K^{\alpha}{}_{\alpha}{}^{\kappa}B_{\kappa;}{}^{\varepsilon}{}_{\varepsilon} + 15K^{\alpha}{}_{\alpha}{}^{\kappa}B^{\varepsilon}{}_{;\varepsilon\kappa} + 15K^{\alpha}{}_{\alpha}{}^{\kappa}K^{\varepsilon}{}_{\varepsilon}{}^{\rho}{}_{;\kappa}B_{\rho} \\
&\quad + 15K^{\alpha}{}_{\alpha}{}^{\kappa}K^{\varepsilon}{}_{\varepsilon}{}^{\rho}B_{\rho;\kappa} - \tfrac{15}{2}B^{\kappa}{}_{;\varepsilon}{}^{\varepsilon}B_{\kappa} - 15K^{\kappa\varepsilon\rho}B_{\rho;\varepsilon}B_{\kappa} - \tfrac{15}{2}K^{\varepsilon}{}_{\varepsilon}{}^{\rho}B^{\kappa}{}_{;\rho}B_{\kappa} \\
&\quad - \tfrac{15}{2}B^{\kappa}B_{\kappa;}{}^{\varepsilon}{}_{\varepsilon} - 15K^{\kappa\varepsilon\rho}B_{\kappa}B_{\rho;\varepsilon} - \tfrac{15}{2}K^{\varepsilon}{}_{\varepsilon}{}^{\rho}B_{\kappa}B^{\kappa}{}_{;\rho} \\
&\quad + 15B^{\kappa}{}_{;\kappa\varepsilon}B^{\varepsilon} + 15K^{\alpha}{}_{\alpha}{}^{\kappa}{}_{;\varepsilon}B_{\kappa}B^{\varepsilon} + 15K^{\alpha}{}_{\alpha}{}^{\kappa}B_{\kappa;\varepsilon}B^{\varepsilon} - 15B^{\varepsilon}B^{\kappa}{}_{;\kappa\varepsilon} \\
&\quad - 15K^{\alpha}{}_{\alpha}{}^{\kappa}{}_{;\varepsilon}B^{\varepsilon}B_{\kappa} - 15K^{\alpha}{}_{\alpha}{}^{\kappa}B^{\varepsilon}B_{\kappa;\varepsilon} + \tfrac{45}{2}B^{\kappa}{}_{;\kappa}B^{\varepsilon}{}_{;\varepsilon} + \tfrac{45}{2}K^{\alpha}{}_{\alpha}{}^{\varepsilon}B^{\kappa}{}_{;\kappa}B_{\varepsilon} \\
&\quad + \tfrac{45}{2}K^{\alpha}{}_{\alpha}{}^{\varepsilon}B_{\varepsilon}B^{\kappa}{}_{;\kappa} + \tfrac{45}{2}K^{\varepsilon}{}_{\varepsilon}{}^{\kappa}K^{\alpha}{}_{\alpha}{}^{\rho}B_{\kappa}B_{\rho} - \tfrac{15}{2}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\kappa;\varepsilon} - \tfrac{15}{2}K^{\kappa\varepsilon\rho}B_{\kappa;\varepsilon}B_{\rho} \\
&\quad - \tfrac{15}{2}K^{\kappa\varepsilon\rho}B_{\rho}B_{\kappa;\varepsilon} - \tfrac{15}{2}K^{\alpha\beta\kappa}K_{\alpha\beta}{}^{\varepsilon}B_{\kappa}B_{\varepsilon} - \tfrac{15}{2}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\varepsilon;\kappa} - \tfrac{15}{2}K^{\varepsilon\kappa\rho}B_{\kappa;\varepsilon}B_{\rho} \\
&\quad - \tfrac{15}{2}K^{\varepsilon\kappa\rho}B_{\kappa;\varepsilon}B_{\rho} - \tfrac{15}{2}K^{\kappa\varepsilon\rho}B_{\rho}B_{\varepsilon;\kappa} - \tfrac{15}{2}K^{\alpha\beta\kappa}K_{\beta\alpha}{}^{\varepsilon}B_{\kappa}B_{\varepsilon} \\
&\quad - \tfrac{15}{4}B^{\kappa}{}_{;\kappa}B^{\varepsilon}B_{\varepsilon} - \tfrac{15}{4}K^{\alpha}{}_{\alpha}{}^{\kappa}B_{\kappa}B^{\varepsilon}B_{\varepsilon} - \tfrac{15}{4}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\kappa}B_{\varepsilon} - \tfrac{15}{4}B^{\kappa}{}_{;}{}^{\varepsilon}B_{\varepsilon}B_{\kappa} + \tfrac{15}{4}B^{\kappa}B^{\varepsilon}B_{\kappa;\varepsilon} \\
&\quad + \tfrac{15}{2}K^{\kappa\varepsilon\rho}B_{\kappa}B_{\varepsilon}B_{\rho} + \tfrac{15}{4}B^{\kappa}B^{\varepsilon}B_{\varepsilon;\kappa} - \tfrac{15}{2}B^{\kappa}B_{\kappa;\varepsilon}B^{\varepsilon} - \tfrac{15}{4}K^{\kappa\varepsilon\rho}B_{\kappa}B_{\rho}B_{\varepsilon} + \tfrac{15}{2}B^{\kappa}B_{\varepsilon;\kappa}B^{\varepsilon} \\
&\quad + \tfrac{15}{4}K^{\varepsilon\kappa\rho}B_{\kappa}B_{\rho}B_{\varepsilon} - \tfrac{45}{4}B^{\varepsilon}B_{\varepsilon}B^{\kappa}{}_{;\kappa} - \tfrac{45}{4}K^{\alpha}{}_{\alpha}{}^{\kappa}B^{\varepsilon}B_{\varepsilon}B_{\kappa} - \tfrac{15}{2}B^{\varepsilon}B^{\kappa}{}_{;\kappa}B_{\varepsilon} \\
&\quad - \tfrac{15}{2}K^{\alpha}{}_{\alpha}{}^{\kappa}B^{\varepsilon}B_{\kappa}B_{\varepsilon} + \tfrac{15}{8}B^{\kappa}B_{\kappa}B^{\varepsilon}B_{\varepsilon} + \tfrac{15}{8}B^{\kappa}B^{\varepsilon}B_{\kappa}B_{\varepsilon} + \tfrac{15}{8}B^{\kappa}B^{\varepsilon}B_{\varepsilon}B_{\kappa}\}
\end{aligned} \tag{4.14}$$

where $B^{\kappa} = (\tilde{B}^{\kappa} + K^{\varepsilon}{}_{\varepsilon}{}^{\kappa})$ and the tensors $M_{\alpha\beta\gamma\delta}$, $M_{\alpha\beta}$ and M are as in appendix 2.

When the torsion is totally antisymmetric things are simpler. Equations (4.12) simplify, as do the relations for $M_{\alpha\beta\kappa\epsilon}$ and its contractions. One obtains from (4.11):

$$
\begin{aligned}
&180[a_2(-\tilde{\Box}+\tilde{B}^\kappa\tilde{\nabla}_\kappa+X)]\\
&\quad=\{\tilde{R}_{\alpha\beta\gamma\delta}\tilde{R}^{\alpha\beta\gamma\delta}-\tilde{R}_{\alpha\beta}\tilde{R}^{\alpha\beta}+\tfrac{5}{2}\tilde{R}^2+6\tilde{R}_{;\kappa}{}^{\kappa}+2\tilde{R}^{\alpha\beta\kappa\epsilon}K_{\alpha\beta\kappa;\epsilon}-2\tilde{R}^{\alpha\beta\kappa\epsilon}K_{\alpha\kappa\rho}K_{\beta\epsilon}{}^{\rho}\\
&\qquad+2\tilde{R}^{\alpha\beta\kappa\epsilon}K_{\alpha\beta\rho}K_{\kappa\epsilon}{}^{\rho}+8K_{\alpha\beta\kappa;\epsilon}K^{\alpha\beta\kappa}{}_{;}{}^{\epsilon}-2K_{\alpha\beta\kappa;\epsilon}K^{\alpha\beta\epsilon}{}_{;}{}^{\kappa}+8K^{\alpha\beta\kappa}{}_{;}{}^{\epsilon}K_{\alpha\beta}{}^{\rho}K_{\kappa\epsilon\rho}\\
&\qquad-8K^{\alpha\beta\kappa}{}_{;}{}^{\epsilon}K_{\alpha\kappa\rho}K_{\beta\epsilon}{}^{\rho}+5K^{\alpha\beta\kappa}K_{\alpha\beta}{}^{\epsilon}K_{\kappa}{}^{\mu\nu}K_{\epsilon\mu\nu}-10K^{\alpha\beta\kappa}K^{\mu\nu}{}_{\kappa}K_{\alpha\mu}{}^{\epsilon}K_{\beta\nu\epsilon}\\
&\qquad+\tilde{R}^{\alpha\beta}K_{\alpha\beta\kappa;}{}^{\kappa}-2\tilde{R}^{\alpha\beta}K_{\alpha}{}^{\kappa\epsilon}K_{\beta\kappa\epsilon}+5\tilde{R}K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}+\tfrac{5}{2}(K_{\alpha\beta\kappa}K^{\alpha\beta\kappa})^2\\
&\qquad+12K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}{}_{;\epsilon}{}^{\epsilon}-30\tilde{R}X-30K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}X-30X_{;\kappa}{}^{\kappa}+90X^2\\
&\qquad+15Y_{\kappa\epsilon}Y^{\kappa\epsilon}+\tfrac{15}{2}\tilde{B}^\kappa\tilde{B}^\epsilon Y_{\kappa\epsilon}+\tfrac{15}{2}Y_{\kappa\epsilon}\tilde{B}^\kappa\tilde{B}^\epsilon+\tfrac{15}{2}\tilde{B}^\kappa Y_{\kappa\epsilon}\tilde{B}^\epsilon+30\tilde{B}^\kappa{}_{;}{}^{\epsilon}Y_{\kappa\epsilon}\\
&\qquad+30K^{\kappa\epsilon\rho}\tilde{B}_\rho Y_{\kappa\epsilon}+15Y_{\kappa\epsilon}\tilde{B}^\kappa{}_{;}{}^{\epsilon}+15Y_{\kappa\epsilon}K^{\kappa\epsilon\rho}\tilde{B}_\rho\\
&\qquad+\tfrac{15}{2}\tilde{B}^\kappa Y_{\kappa\epsilon;}{}^{\epsilon}+\tfrac{15}{2}K^{\kappa\epsilon\rho}\tilde{B}_\kappa Y_{\rho\epsilon}+15\tilde{R}\tilde{B}^\kappa{}_{;\kappa}+15K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}\tilde{B}^\kappa{}_{;\kappa}-\tfrac{15}{2}\tilde{R}\tilde{B}^\kappa\tilde{B}_\kappa\\
&\qquad-\tfrac{15}{2}K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}\tilde{B}^\epsilon\tilde{B}_\epsilon+15\tilde{B}^\kappa\tilde{B}_\kappa X+15X\tilde{B}^\kappa\tilde{B}_\kappa+15\tilde{B}^\kappa X\tilde{B}_\kappa\\
&\qquad-30\tilde{B}^\kappa{}_{;\kappa}X-60X\tilde{B}^\kappa{}_{;\kappa}+30\tilde{B}^\kappa X_{;\kappa}-30X_{;\kappa}\tilde{B}^\kappa+15\tilde{B}^\kappa{}_{;\kappa}{}^{\epsilon}{}_{\epsilon}-\tfrac{15}{2}\tilde{B}^\kappa{}_{;\epsilon}{}^{\epsilon}\tilde{B}_\kappa\\
&\qquad-15K^{\kappa\epsilon\rho}\tilde{B}_{\rho;\epsilon}\tilde{B}_\kappa-\tfrac{15}{2}\tilde{B}^\kappa\tilde{B}_{\kappa;}{}^{\epsilon}{}_{\epsilon}-15K^{\kappa\epsilon\rho}\tilde{B}_\kappa\tilde{B}_{\rho;\epsilon}\\
&\qquad+15\tilde{B}^\kappa{}_{;\kappa\epsilon}\tilde{B}^\epsilon-15\tilde{B}^\epsilon\tilde{B}^\kappa{}_{;\kappa\epsilon}+\tfrac{45}{2}\tilde{B}^\kappa{}_{;\kappa}\tilde{B}^\epsilon{}_{;\epsilon}-\tfrac{15}{2}\tilde{B}^\kappa{}_{;}{}^{\epsilon}\tilde{B}_{\kappa;\epsilon}-\tfrac{15}{2}\tilde{B}^\kappa{}_{;}{}^{\epsilon}\tilde{B}_{\epsilon;\kappa}\\
&\qquad-\tfrac{15}{4}\tilde{B}^\kappa{}_{;\kappa}\tilde{B}^\epsilon\tilde{B}_\epsilon-\tfrac{15}{4}\tilde{B}^\kappa{}_{;}{}^{\epsilon}\tilde{B}_\kappa\tilde{B}_\epsilon-\tfrac{15}{4}\tilde{B}^\kappa{}_{;}{}^{\epsilon}\tilde{B}_\epsilon\tilde{B}_\kappa+\tfrac{15}{4}\tilde{B}^\kappa\tilde{B}^\epsilon\tilde{B}_{\kappa;\epsilon}+15K^{\kappa\epsilon\rho}\tilde{B}_\kappa\tilde{B}_\epsilon\tilde{B}_\rho\\
&\qquad+\tfrac{15}{4}\tilde{B}^\kappa\tilde{B}^\epsilon\tilde{B}_{\epsilon;\kappa}-\tfrac{15}{2}\tilde{B}^\kappa\tilde{B}_{\kappa;\epsilon}\tilde{B}^\epsilon+\tfrac{15}{2}\tilde{B}^\kappa\tilde{B}_{\epsilon;\kappa}\tilde{B}^\epsilon-\tfrac{45}{4}\tilde{B}^\epsilon\tilde{B}_\epsilon\tilde{B}^\kappa{}_{;\kappa}-\tfrac{15}{2}\tilde{B}^\epsilon\tilde{B}^\kappa{}_{;\kappa}\tilde{B}_\epsilon\\
&\qquad+\tfrac{15}{8}\tilde{B}^\kappa\tilde{B}_\kappa\tilde{B}^\epsilon\tilde{B}_\epsilon+\tfrac{15}{8}\tilde{B}^\kappa\tilde{B}^\epsilon\tilde{B}_\kappa\tilde{B}_\epsilon+\tfrac{15}{8}\tilde{B}^\kappa\tilde{B}^\epsilon\tilde{B}_\epsilon\tilde{B}_\kappa\}.
\end{aligned}
\tag{4.15}
$$

When the tensor B^κ and X commute as they do in (4.12) one obtains from the totally antisymmetric torsion case (4.15) above

$$
\begin{aligned}
&180[a_2(-\tilde{\Box}+\tilde{B}^\kappa\tilde{\nabla}_\kappa+X)]\\
&\quad=\{\tilde{R}_{\alpha\beta\gamma\delta}\tilde{R}^{\alpha\beta\gamma\delta}-\tilde{R}_{\alpha\beta}\tilde{R}^{\alpha\beta}+\tfrac{5}{2}\tilde{R}^2+6\tilde{R}_{;\kappa}{}^{\kappa}+2\tilde{R}^{\alpha\beta\kappa\epsilon}K_{\alpha\beta\kappa;\epsilon}-2\tilde{R}^{\alpha\beta\kappa\epsilon}K_{\alpha\kappa\rho}K_{\beta\epsilon}{}^{\rho}\\
&\qquad+2\tilde{R}^{\alpha\beta\kappa\epsilon}K_{\alpha\beta\rho}K_{\kappa\epsilon}{}^{\rho}+8K_{\alpha\beta\kappa;\epsilon}K^{\alpha\beta\kappa}{}_{;}{}^{\epsilon}-2K_{\alpha\beta\kappa;\epsilon}K^{\alpha\beta\epsilon}{}_{;}{}^{\kappa}+8K^{\alpha\beta\kappa}{}_{;}{}^{\epsilon}K_{\alpha\beta}{}^{\rho}K_{\kappa\epsilon\rho}\\
&\qquad-8K^{\alpha\beta\kappa}{}_{;}{}^{\epsilon}K_{\alpha\kappa\rho}K_{\beta\epsilon}{}^{\rho}+5K^{\alpha\beta\kappa}K_{\alpha\beta}{}^{\epsilon}K_{\kappa}{}^{\mu\nu}K_{\epsilon\mu\nu}-10K^{\alpha\beta\kappa}K^{\mu\nu}{}_{\kappa}K_{\alpha\mu}{}^{\epsilon}K_{\beta\nu\epsilon}\\
&\qquad+\tilde{R}^{\alpha\beta}K_{\alpha\beta\kappa;}{}^{\kappa}-2\tilde{R}^{\alpha\beta}K_{\alpha}{}^{\kappa\epsilon}K_{\beta\kappa\epsilon}+5\tilde{R}K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}+\tfrac{5}{2}(K_{\alpha\beta\kappa}K^{\alpha\beta\kappa})^2\\
&\qquad+12K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}{}_{;\epsilon}{}^{\epsilon}-30\tilde{R}X-30K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}X-30X_{;\kappa}{}^{\kappa}+90X^2\\
&\qquad+15Y_{\kappa\epsilon}Y^{\kappa\epsilon}+45\tilde{B}^\kappa{}_{;}{}^{\epsilon}Y_{\kappa\epsilon}+\tfrac{75}{2}K^{\kappa\epsilon\rho}Y_{\kappa\epsilon}\tilde{B}_\rho+\tfrac{15}{2}\tilde{B}^\kappa Y_{\kappa\epsilon;}{}^{\epsilon}+15\tilde{R}\tilde{B}^\kappa{}_{;\kappa}\\
&\qquad+15K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}\tilde{B}_{\epsilon;}{}^{\epsilon}-\tfrac{15}{2}\tilde{R}\tilde{B}^\kappa\tilde{B}_\kappa-\tfrac{15}{2}K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}\tilde{B}^\epsilon\tilde{B}_\epsilon\\
&\qquad+45\tilde{B}^\kappa\tilde{B}_\kappa X-90\tilde{B}^\kappa{}_{;\kappa}X+15\tilde{B}^\kappa{}_{;\kappa}{}^{\epsilon}{}_{\epsilon}-15\tilde{B}^\kappa\tilde{B}_{\kappa;}{}^{\epsilon}{}_{\epsilon}+30K^{\alpha\beta\kappa}\tilde{B}_{\alpha;\beta}\tilde{B}_\kappa\\
&\qquad+\tfrac{45}{2}\tilde{B}^\kappa{}_{;\kappa}\tilde{B}^\epsilon{}_{;\epsilon}-\tfrac{15}{2}\tilde{B}^\kappa{}_{;}{}^{\epsilon}\tilde{B}_{\kappa;\epsilon}-\tfrac{15}{2}\tilde{B}^\kappa{}_{;}{}^{\epsilon}\tilde{B}_{\epsilon;\kappa}-\tfrac{45}{2}\tilde{B}^\kappa\tilde{B}_\kappa\tilde{B}^\epsilon{}_{;\epsilon}+\tfrac{45}{8}\tilde{B}^\kappa\tilde{B}_\kappa\tilde{B}^\epsilon\tilde{B}_\epsilon\}.
\end{aligned}
\tag{4.16}
$$

5. Comparison with a direct calculation

It would be nice to check the work of the previous section with a direct calculation. For reasons already mentioned, the full direct calculation of (4.13) is a formidable task likely to involve human error. We shall discuss previous work taking the direct aporoach in the next section. For now we propose the modest direct calculation of $[a_2(-\tilde{\Box}+X)]$, using a severely restricted torsion tensor and Riemann curvature which will make the calculation possible.

Consider a semisimple Lie group and connection $\tilde{\Gamma}$ as in theorem 2.6 and proposition 2.12 in ch 10 of Kobayashi and Nomizu (1969). It then follows that there exists a non-zero torsion tensor which is totally antisymmetric and covariantly constant and a curvature tensor such that $\tilde{R}_{\alpha\beta\kappa\varepsilon}=0$ (we thank M Bordermann and M Forger for bringing this example to our attention). Using these restrictions there is the important identity which follows immediately from the cyclic identity (A2.17):

$$T_{\alpha\beta\kappa}T^{\alpha\beta}{}_{\varepsilon}T_{\mu\nu}{}^{\kappa}T^{\mu\nu\varepsilon}=2T^{\alpha\beta\kappa}T^{\mu\nu}{}_{\kappa}T_{\alpha\mu}{}^{\varepsilon}T_{\beta\nu\varepsilon}. \tag{5.1}$$

As an aside, we mention that it is possible to show that in general a totally antisymmetric and covariantly constant torsion tensor is not zero. In three dimensions any totally antisymmetric tensor must be proportional to the Levi-Civita:

$$T_{[ijk]}\sim\varepsilon_{ijk} \tag{5.2}$$

where here i, j and $k=1, 2, 3$. Then, given some constant of proportionality A with the appropriate units, it follows from the covariant constancy of the Levi-Civita that

$$\overset{3}{\tilde{\nabla}}_l T_{[ijk]}=A\overset{3}{\tilde{\nabla}}_l\varepsilon_{ijk}=0 \tag{5.3}$$

where $\overset{3}{\tilde{\nabla}}_l$ is the torsionful three-dimensional covariant derivative. Thus the tensor T_{ijk} has all of the desired properties and is also non-zero by construction. It is easy to see that T_{ijk} is part of $T_{\alpha\beta\kappa}$ and therefore we have shown what we set out to show.

Moving on, using the restrictions on $\tilde{R}_{\alpha\beta\kappa\varepsilon}$ and $T_{\alpha\beta\kappa}$, and (5.1), equation (4.13) becomes simply

$$180[a_2(-\tilde{\Box}+X)]=\{\tfrac{5}{2}(K_{\alpha\beta\kappa}K^{\alpha\beta\kappa})^2-30K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}X$$
$$-30\tilde{\Box}X+90X^2+15Y_{\mu\nu}Y^{\mu\nu}\}. \tag{5.4}$$

We now obtain this directly. Using the ansatz (3.5) as we did in § 4, we obtain for the operator $(-\Box+X)$ the indicial equation

$$\{-\tfrac{1}{2}na_0+\sigma_{;}{}^{\kappa}a_{0;\kappa}+\tfrac{1}{2}\sigma_{;\kappa}{}^{\kappa}a_0\}=0 \tag{5.5}$$

and recursion relations

$$\{-\tfrac{1}{2}na_{l+1}+(l+1)a_{l+1}+\sigma_{;}{}^{\kappa}a_{l+1;\kappa}+\tfrac{1}{2}\sigma_{;\kappa}{}^{\kappa}a_{l+1}-a_{l;\kappa}{}^{\kappa}+Xa_l\}=0 \tag{5.6}$$

(ignoring the indices i, j and k for now). By taking derivatives of these two equations, using the generalised Ricci identity (2.15) and coincidence limits from appendix 3, we obtain

$$[a_{0;\kappa}]=0 \tag{5.7a}$$

$$[a_{0;\alpha\beta}]=\{-\tfrac{1}{4}[\sigma_{;\kappa}{}^{\kappa}{}_{\alpha\beta}]-\tfrac{1}{2}Y_{\alpha\beta}\} \tag{5.7b}$$

$$[\sigma_{;}{}^{\alpha\beta\kappa}][a_{0;\alpha\beta\kappa}]=\tfrac{1}{6}|[\sigma_{;\alpha\beta\kappa}][\sigma_{;}{}^{\alpha\beta}{}_{\varepsilon}]|^2 \tag{5.7c}$$

$$[a_{0;\kappa}{}^{\kappa}{}_{\varepsilon}{}^{\varepsilon}]=\{\tfrac{1}{2}Y_{\mu\nu}Y^{\mu\nu}+\tfrac{1}{36}([\sigma_{;\alpha\beta\kappa}][\sigma_{;}{}^{\alpha\beta\kappa}])^2\}. \tag{5.7d}$$

Then after a little more work one obtains

$$[a_1]=\{\tfrac{1}{6}K_{\alpha\beta\kappa}K^{\alpha\beta\kappa}-X\} \tag{5.8}$$

and equation (5.4) for the a_2 coefficient of $(-\tilde{\Box}+X)$. Thus (5.4) has been confirmed by direct calculation.

6. Discussion

Equations (4.10) and (4.11) are somewhat curious because not all of the possible invariants that could appear in them do appear. In (4.10) the following invariants do not appear:

$$B^{\kappa}R_{;\kappa} \qquad B^{\kappa}B^{\epsilon}R_{\kappa\epsilon} \qquad B^{\kappa}{}_{;}{}^{\epsilon}R_{\kappa\epsilon} \qquad Y_{\kappa\epsilon;}{}^{\epsilon}B^{\kappa}. \tag{6.1}$$

Similarly, in (4.11) the 'missing' invariants are

$$B^{\kappa}R_{;\kappa} \qquad B^{\kappa}X_{,\kappa} \qquad B^{\kappa}B^{\epsilon}R_{\kappa\epsilon} \qquad B^{\kappa}{}_{;}{}^{\epsilon}R_{\kappa\epsilon} \qquad B^{\kappa}{}_{;\kappa\epsilon}B^{\epsilon} \qquad B^{\kappa}{}_{;}{}^{\epsilon}B_{\kappa}B_{\epsilon}. \tag{6.2}$$

This is, as far as we know, the first time that this has been 'observed'. The expectation that all possible invariants that can appear *do* appear in the asymptotic expansion coefficients has been used, for example, to argue that ordinary pure gravity theories are hopelessly non-renormalisable. At every loop one expects every possible invariant to appear which, due to dimensional reasons, can never be absorbed by adjustment of the coupling constants into the action. One is forced to add an infinite number of counterterms to the action. Thus it is natural to ask whether the Schwinger-DeWitt ansatz works for the operator $(-\Box + B^{\kappa}\nabla_{\kappa} + X)$. There are second-order operators for which the ansatz does not work such as (6.3) below (as well as any operator of order $\geqslant 4$).

The expression for $[a_2(-\Box + X)]$ obtained via the ansatz is known to agree with other calculational methods (DeWitt 1965, 1975, Gilkey 1975). As this is true, the ansatz must also work for the operator $(-\tilde{\Box} - K^{\epsilon}{}_{\epsilon}{}^{\kappa}\tilde{\nabla}_{\kappa} + X)$ (in fact using (2.13) it is easy to transform the indicial equation and recursion relations for the one operator into those of the other). When this is true, then it must also be the case that the ansatz works for the generalisation of this operator: $(-\tilde{\Box} + \tilde{B}^{\kappa}\tilde{\nabla}_{\kappa} + X)$. Simply taking the limiting case of when the torsion goes to zero in $[a_2(-\tilde{\Box} + \tilde{B}^{\kappa}\tilde{\nabla}_{\kappa} + X)]$ then gives $[a_2(-\Box + B^{\kappa}\nabla_{\kappa} + X)]$. So it seems that the ansatz is applicable to the operators we considered in § 4.

There has been other work on finding $[a_2(-\tilde{\Box} + \tilde{B}^{\kappa}\tilde{\nabla}_{\kappa} + X)]$. Goldthorpe (1980) and Nieh and Yan (1982) have attempted to perform the direct calculation using the Schwinger-DeWitt ansatz. Goldthorpe reduces the complexity by requiring that the torsion be totally antisymmetric, rather than completely arbitrary. If we take the zero torsion limit of Goldthorpe's expression for $[a_2(-\tilde{\Box} + \tilde{B}^{\kappa}\tilde{\nabla}_{\kappa} + X)]$ we find that his results do not agree with either our (4.10) or (4.11). One instance is that the invariant $\tilde{B}^{\kappa}{}_{;\kappa}{}^{\epsilon}{}_{\epsilon}$ is missing in his expression. It is possible to trace this error through to his equation (A5) which is not consistent with his (3.7). There are other discrepancies with our results as well in this limiting case. We also note that Goldthorpe's results are not in their most simple form. Many of his invariants are related to each other via various identities.

Nieh and Yan give a direct calculation for the general case of $[a_2(-\tilde{\Box} + \tilde{B}^{\kappa}\tilde{\nabla}_{\kappa} + X)]$ with arbitrary torsion. They are unable to give their results in closed form leaving many significant substitutions to the reader. Indeed, in some ways their calculation is left incomplete. For this reason it is rather hard to work with their results. However, taking the limit of zero torsion, we obtain discrepancies with our results for $[a_2(-\Box + B^{\kappa}\nabla_{\kappa} + X)]$ given in (4.10) and (4.11).

There is also the work of Obukhov (1982, 1983). He also suggests that there might be a relationship between a_2 coefficients for various operators. As we have, he tried to use this to find $[a_2(-\tilde{\Box} + \tilde{B}^{\kappa}\tilde{\nabla}_{\kappa} + X)]$ from non-torsion cases. He correctly claims that it is possible to obtain $[a_2(-\tilde{\Box}+\tilde{B}^\kappa\tilde{\nabla}_\kappa+X)]$ from $[a_2(-\Box+B^\kappa\nabla_\kappa+X)]$, but then goes on to claim that $[a_2(-\Box+B^\kappa\nabla_\kappa+X)]$ can, in the same fashion, be obtained from $[a_2(-\Box+X)]$. He is not alone in this. It is quite commonly suggested that this is possible by simply introducing another connection and derivative (see e.g. Barvinsky and Vilkovisky (1985)). What most seem to fail to realise however is that not every derivative is metric compatible. If one demands metric compatibility then the new derivatives turn out to have torsion in them. This is what we showed in § 3 and this is what equation (3.24) and the primed derivatives are all about. Clearly if one is trying to solve for the torsion case to begin with then introducing these primed derivatives is of no help. If one is willing to live with non-metric compatible derivatives one is perhaps able to rewrite the operator $(-\Box+B^\kappa\nabla_\kappa+X)$ in the form $(-A^{\alpha\beta}\nabla_\alpha\nabla_\beta+X)$. However the a_2 coefficient for operators of this form is not known, even when $A^{\alpha\beta}$ equals the metric for non-metric compatible derivatives. In short we do believe it is possible to obtain $[a_2(-\Box+B^\kappa\nabla_\kappa+X)]$ from $[a_2(-\Box+X)]$ in general. Having said all of this, we note that Obukhov's expressions do not take the correct limiting form of (4.11) when the torsion goes to zero in $[a_2(-\tilde{\Box}+\tilde{B}^\kappa\tilde{\nabla}_\kappa+X)]$ (he does not seem to deal with the more general case of when $[B^\kappa, B^\epsilon]\neq 0$ etc, which has a limiting case given by (4.10)). Lastly, in the various proofs and arguments given by Obukhov (1982, 1983), he never mentions that it is due to the fact that Δ_1 and Δ_2 act on the same set of eigenfunctions in the same way, and thus have the *same* spectrum, that their asymptotic expansions are equivalent. In fact as far as the a_2 coefficient is concerned one really only needs to show that the *number* of zero and non-zero modes for these operators are the same. We consider these to be the key points allowing the a_2 coefficients of Δ_1 and Δ_2 (and the rest of the asymptotic expansion coefficients) to be related to each other.

We would like to make an additional comment on expressions for the a_2 coefficient for operators of the form

$$\{-A^{(\alpha\beta)}\nabla_\alpha\nabla_\beta+B^\kappa\nabla_\kappa+C\}. \tag{6.3}$$

This is the most general elliptical operator possible with up to two derivatives. Christensen (1982) suggests a method for obtaining the a_2 coefficient for such an operator. As it is presented however, this method can only work for $A^{\alpha\beta}$ such that it is of zeroth order. It should be noted that there is no general expression for a_2 when $A^{\alpha\beta}$ is of arbitrary order. This is because when $A^{\alpha\beta}$ is such that its order is greater than four, the a_2 coefficient must be, on dimensional grounds, completely independent of it, and thus it cannot even appear in the expression for a_2.

We have shown that when calculating a_2 coefficients it is sometimes possible to reduce the operator in question to a simpler operator for which the a_2 coefficient is already known. In § 3 we also showed that it is possible to do away with the middle term in the operator $(-\Box+B^\kappa\nabla_\kappa+X)$ by introducing the primed derivatives and the operator $(-\tilde{\Box}+X)$ (or $(-\tilde{\tilde{\Box}}+X)$). Is this possible in general? That is, given the general operator of order $n=2k$ with the leading symbol given by some power of the metric

$$\Delta_n=\{(-1)^k\Box^k+A^{(\alpha_1\ldots\alpha_{n-1})}\nabla_{\alpha_1}\ldots\nabla_{\alpha_{n-1}}+\ldots+A^{\alpha_1}\nabla_{\alpha_1}+A\} \tag{6.4}$$

does there always exist an operator

$$\bar{\Delta}_n=\{(-1)^k\bar{\Box}^k+\bar{A}^{(\alpha_1\ldots\alpha_{n-2})}\bar{\nabla}_{\alpha_1}\ldots\bar{\nabla}_{\alpha_{n-2}}+\ldots+\bar{A}^{\alpha_1}\bar{\nabla}_{\alpha_1}+\bar{A}\} \tag{6.5}$$

such that they act on eigenfunctions in the same way? Although we have shown this to be true for the case $n=2$ this does not seem possible in general.

Areas for further investigation would be to calculate anomalies using the results here. Preliminary attempts can be found in Obukhov (1983) and Yajima and Kimura (1985) but they use algorithms found in Obukhov (1983) and Nieh and Yan (1982) respectively for $[a_2(-\tilde{\Box}+\tilde{B}_\kappa\tilde{\nabla}_\kappa+X)]$ which do not satisfy the limiting cases of (4.10) or (4.11). The work of Yajima and Kimura (1985) also considers only the special case of totally antisymmetric torsion.

It is also now possible, in principle, using the results here and in Barth and Christensen (1983), to do for the torsion case what Christensen and Duff (1979) did for the torsionless case. This would be to catalogue the relationships between spin, a_2 coefficients, index theorems and anomalies. It then might be possible to find various combinations of spin fields which give rise to interesting cancellations.

It would also be useful to investigate why certain invariants given in (6.1) and (6.2) do not appear in (4.10) or (4.11) respectively. A deeper understanding of this could be significant for calculating these asymptotic coefficients. We shall return to this when we consider higher-order operators in the second paper of this series.

Lastly, it would be useful to try to calculate $[a_2(-\tilde{\Box}+\tilde{B}^\kappa\tilde{\nabla}_\kappa+X)]$ directly using a computer. Although, as we have suggested, the direct calculation is difficult for humans, a computer has obvious advantages. As our results seem to be the most consistent with the various limiting cases they might be useful for comparison with computer results. The calculation is of such significant complexity that it might also be a good test for a symbolic manipulation program.

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Appendix 1. Coincidence limits of the geodetic interval

Taking derivatives of the identity

$$\sigma(x,x')=\tfrac{1}{2}\sigma(x,x')_{;\kappa}\sigma(x,x')_{;}{}^{\kappa} \tag{A1.1}$$

and then, after use of the Ricci identity, the coincidence limit as $x'\to x$ (denoted by brackets), one obtains the relations

$$[\sigma(x,x')]\equiv 0 \tag{A1.2}$$

$$[\sigma_{;\kappa}]=0 \tag{A1.3}$$

$$[\sigma_{;\kappa\epsilon}]=g_{\kappa\epsilon} \tag{A1.4}$$

$$[\sigma_{,\alpha\beta\kappa}]=0 \tag{A1.5}$$

$$[\sigma_{;\alpha\beta\mu\nu}]=-\tfrac{1}{3}\{R_{\alpha\mu\beta\nu}+R_{\alpha\nu\beta\mu}\} \tag{A1.6a}$$

$$[\sigma_{;\alpha\beta\kappa}{}^{\kappa}]=[\sigma_{;\kappa}{}^{\kappa}{}_{\alpha\beta}]=-\tfrac{2}{3}R_{\alpha\beta} \tag{A1.6b}$$

$$[\sigma_{;\alpha\kappa}{}^{\kappa}{}_{\beta}]=\tfrac{1}{3}R_{\alpha\beta} \tag{A1.6c}$$

$$|[\sigma_{;\alpha\beta\mu\nu}]|^2=\tfrac{1}{3}|R_{\alpha\beta\mu\nu}|^2=\tfrac{1}{3}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \tag{A1.6d}$$

$$[\sigma_{;\alpha\beta\mu\nu\kappa}] = -\tfrac{1}{4}\{R_{\alpha\mu\beta\nu;\kappa} + R_{\alpha\kappa\beta\nu;\mu} + R_{\alpha\nu\beta\mu;\kappa} + R_{\alpha\kappa\beta\mu;\nu} + R_{\alpha\mu\beta\kappa;\nu} + R_{\alpha\nu\beta\kappa;\mu}\} \quad (A1.7a)$$

$$[\sigma_{;\kappa}{}^{\kappa}{}_{\alpha}{}^{\alpha}{}_{\epsilon}] = [\sigma_{;\kappa}{}^{\kappa}{}_{\epsilon\alpha}{}^{\alpha}] = -R_{;\epsilon} \quad (A1.7b)$$

$$[\sigma_{;\epsilon\kappa}{}^{\kappa}{}_{\alpha}{}^{\alpha}] = 0 \quad (A1.7c)$$

$$[\sigma_{;\kappa}{}^{\kappa}{}_{\epsilon}{}^{\epsilon}{}_{\alpha}{}^{\alpha}] = \{-\tfrac{4}{15}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} + \tfrac{4}{15}R_{\alpha\beta}R^{\alpha\beta} - \tfrac{8}{5}R_{;\kappa}{}^{\kappa}\}. \quad (A1.8)$$

Appendix 2. Riemann, connection and torsion relations

Given

$$(-\tilde{\square} + \tilde{B}^{\kappa}\tilde{\nabla}_{\kappa})\phi = (-\square + B^{\kappa}\nabla_{\kappa})\phi = -\acute{\square}\phi = -\acute{\acute{\square}}\phi \quad (A2.1)$$

then the Riemann tensor $R_{\alpha\beta\mu\nu}$ is built out of symmetric connections $\Gamma^{\kappa}{}_{\alpha\beta}$ and $\tilde{R}_{\alpha\beta\mu\nu}$, $\acute{R}_{\alpha\beta\mu\nu}$ and $\acute{\acute{R}}_{\alpha\beta\mu\nu}$ are built out of the antisymmetric connections $\tilde{\Gamma}^{\kappa}{}_{\alpha\beta}$, $\acute{\Gamma}^{\kappa}{}_{\alpha\beta}$ and $\acute{\acute{\Gamma}}^{\kappa}{}_{\alpha\beta}$ respectively, where

$$\tilde{\Gamma}^{\alpha}{}_{\alpha\beta} = \Gamma^{\kappa}{}_{\alpha\beta} + K_{\alpha\beta}{}^{\kappa} \quad (A2.2)$$

$$\acute{\Gamma}^{\kappa}{}_{\alpha\beta} = \Gamma^{\kappa}{}_{\alpha\beta} + \acute{K}_{\alpha\beta}{}^{\kappa} \quad (A2.3)$$

$$\acute{\acute{\Gamma}}^{\kappa}{}_{\alpha\beta} = \tilde{\Gamma}^{\kappa}{}_{\alpha\beta} + \acute{\acute{K}}_{\alpha\beta}{}^{\kappa} \quad (A2.4)$$

$$\acute{K}_{\alpha\beta\kappa} = \tfrac{1}{3}\{g_{\alpha\beta}\tilde{B}_{\kappa} - g_{\kappa\beta}\tilde{B}_{\alpha} + g_{\alpha\beta}K_{\epsilon}{}^{\epsilon}{}_{\alpha} - g_{\kappa\beta}K_{\epsilon}{}^{\epsilon}{}_{\alpha}\} \quad (A2.5a)$$

$$= \tfrac{1}{3}\{g_{\alpha\beta}B_{\kappa} - g_{\kappa\beta}B_{\alpha}\} \quad (A2.5b)$$

$$\acute{\acute{K}}_{\alpha\beta\kappa} = \tfrac{1}{3}\{-3K_{\alpha\beta\kappa} + g_{\alpha\beta}\tilde{B}_{\kappa} - g_{\kappa\beta}\tilde{B}_{\alpha} + g_{\alpha\beta}K_{\epsilon}{}^{\epsilon}{}_{\kappa} - g_{\kappa\beta}K_{\epsilon}{}^{\epsilon}{}_{\alpha}\} \quad (A2.6a)$$

$$= \tfrac{1}{3}\{-3K_{\alpha\beta\kappa} + g_{\alpha\beta}B_{\kappa} - g_{\kappa\beta}B_{\alpha}\} \quad (A2.6b)$$

$$R_{\alpha\beta\mu\nu} = \{\tilde{R}_{\alpha\beta\mu\nu} + K_{\beta\mu\alpha;\nu} - K_{\beta\nu\alpha;\mu} + K_{\beta\nu}{}^{\kappa}K_{\kappa\mu\alpha} - K_{\beta\mu}{}^{\kappa}K_{\kappa\nu\alpha}$$
$$+ K_{\mu\nu}{}^{\kappa}K_{\beta\kappa\alpha} - K_{\nu\mu}{}^{\kappa}K_{\beta\kappa\alpha}\} \quad (A2.7a)$$

$$= \{\tilde{R}_{\alpha\beta\mu\nu} + M_{\alpha\beta\mu\nu}\} \quad (A2.7b)$$

$$R_{\alpha\beta} = \{\tilde{R}_{\alpha\beta} - K_{\alpha\beta\epsilon;}{}^{\epsilon} - K_{\epsilon}{}^{\epsilon}{}_{\alpha;\beta} + K_{\kappa\beta\alpha}K^{\epsilon}{}_{\epsilon}{}^{\kappa} + K_{\epsilon\kappa\alpha}K_{\beta}{}^{\epsilon\kappa}\} \quad (A2.8a)$$

$$= \{\tilde{R}_{\alpha\beta} + M_{\alpha\beta}\} \quad (A2.8b)$$

where

$$M_{\alpha\beta} = M_{\alpha\kappa\beta}{}^{\kappa}$$

$$R = \{\tilde{R} - 2K^{\epsilon}{}_{\epsilon}{}^{\kappa}{}_{;\kappa} + K_{\kappa}{}^{\alpha}{}_{\alpha}K^{\epsilon}{}_{\epsilon}{}^{\kappa} + K_{\epsilon\kappa\alpha}K^{\alpha\epsilon\kappa}\} \quad (A2.9a)$$

$$= \{\tilde{R} + M\} \quad (A2.9b)$$

where

$$M = M_{\kappa}{}^{\kappa} = M_{\kappa\epsilon}{}^{\kappa\epsilon}.$$

Using (2.18) and (A2.3) we obtain

$$\acute{R}_{\alpha\beta\mu\nu} = \{R_{\alpha\beta\mu\nu} + \acute{K}_{\beta\nu\alpha;\mu} - \acute{K}_{\beta\mu\alpha;\nu} + \acute{K}_{\beta\nu}{}^{\kappa}\acute{K}_{\kappa\mu\alpha} - \acute{K}_{\beta\mu}{}^{\kappa}\acute{K}_{\kappa\nu\alpha}\} \quad (A2.10a)$$

$$= \{R_{\alpha\beta\mu\nu} + \acute{M}_{\alpha\beta\mu\nu}\} \quad (A2.10b)$$

$$\acute{R}_{\alpha\beta} = \{R_{\alpha\beta} + \acute{K}_{\epsilon}{}^{\epsilon}{}_{\alpha;\beta} + \acute{K}_{\alpha\beta\epsilon;}{}^{\epsilon} + \acute{K}^{\epsilon}{}_{\epsilon}{}^{\kappa}\acute{K}_{\kappa\beta\alpha} - \acute{K}_{\kappa\epsilon\alpha}\acute{K}^{\epsilon}{}_{\beta}{}^{\kappa}\} \quad (A2.11a)$$

$$= \{R_{\alpha\beta} + \acute{M}_{\alpha\beta}\} \quad (A2.11b)$$

where

$$M'_{\alpha\beta} = M'_{\alpha\kappa\beta}{}^{\kappa}$$

$$R' = \{R + 2K'^{\varepsilon}{}_{\varepsilon}{}^{\kappa}{}_{;\kappa} + K'^{\varepsilon}{}_{\varepsilon}{}^{\kappa}K'_{\kappa}{}^{\alpha}{}_{\alpha} - K'_{\kappa\varepsilon\alpha}K'^{\varepsilon\alpha\kappa}\} \tag{A2.12a}$$

$$= \{R + M'\} \tag{A2.12b}$$

where

$$M' = M'_{\kappa}{}^{\kappa} = M'_{\kappa\varepsilon}{}^{\kappa\varepsilon}.$$

Using (2.8) and (A2.4) we obtain

$$R''_{\alpha\beta\mu\nu} = \{\tilde{R}_{\alpha\beta\mu\nu} + K''_{\beta\nu\alpha;\mu} - K''_{\beta\mu\alpha;\nu} + K''_{\beta\nu}{}^{\kappa}K_{\kappa\mu\alpha} - K''_{\beta\mu}{}^{\kappa}K''_{\kappa\nu\alpha}$$
$$+ K_{\nu\mu}{}^{\kappa}K''_{\beta\kappa\alpha} - K_{\mu\nu}{}^{\kappa}K''_{\beta\kappa\alpha}\}. \tag{A2.13a}$$

Note that the last two terms in the above expression do *not* have primes above the first contorsion tensors. This is as it should be.

$$R''_{\alpha\beta\mu\nu} = \{\tilde{R}_{\alpha\beta\mu\nu} + M''_{\alpha\beta\mu\nu}\} \tag{A2.13b}$$

$$R''_{\alpha\beta} = \{\tilde{R}_{\alpha\beta} + K''_{\kappa}{}^{\kappa}{}_{\alpha;\beta} + K''_{\alpha\beta}{}^{\kappa}{}_{;\kappa} + K''^{\varepsilon}{}_{\varepsilon}{}^{\kappa}K''_{\kappa\beta\alpha} - K''_{\varepsilon\beta}{}^{\kappa}K''_{\kappa}{}^{\varepsilon}{}_{\alpha}$$
$$+ K^{\varepsilon}{}_{\beta}{}^{\kappa}K''_{\varepsilon\kappa\alpha} - K_{\beta}{}^{\varepsilon\kappa}K''_{\varepsilon\kappa\alpha}\} \tag{A2.14a}$$

$$R''_{\alpha\beta} = \{\tilde{R}_{\alpha\beta} + M''_{\alpha\beta}\} \tag{A2.14b}$$

where

$$M''_{\alpha\beta} = M''_{\alpha\kappa\beta}{}^{\kappa}$$

$$R'' = \{\tilde{R} + 2K''^{\kappa}{}_{\kappa}{}^{\varepsilon}{}_{;\varepsilon} + K''^{\varepsilon}{}_{\varepsilon}{}^{\kappa}K''_{\kappa}{}^{\alpha}{}_{\alpha} - K''_{\varepsilon\alpha\kappa}K''^{\kappa\varepsilon\alpha} + 2K_{\varepsilon\alpha\kappa}K''^{\varepsilon\kappa\alpha}\} \tag{A2.15a}$$

$$R'' = \{\tilde{R} + M''\} \tag{A2.15b}$$

where

$$M'' = M''_{\kappa}{}^{\kappa} = M''_{\kappa\varepsilon}{}^{\kappa\varepsilon}.$$

The identities for the torsionful curvature tensor (2.8) follow. The Bianchi identity and contractions are

$$\{\tilde{R}_{\alpha\beta\mu\nu;\kappa} + \tilde{R}_{\alpha\beta\nu\kappa;\mu} + \tilde{R}_{\alpha\beta\kappa\mu;\nu}\} = \{-T_{\mu\nu}{}^{\rho}\tilde{R}_{\alpha\beta\kappa\rho} - T_{\nu\kappa}{}^{\rho}\tilde{R}_{\alpha\beta\mu\rho} - T_{\kappa\mu}{}^{\rho}\tilde{R}_{\alpha\beta\nu\rho}\} \tag{A2.16a}$$

$$\{\tilde{R}_{\alpha\beta;\kappa} - \tilde{R}_{\alpha\kappa;\beta} + \tilde{R}_{\alpha\varepsilon\beta\kappa;}{}^{\varepsilon}\} = \{-T_{\beta}{}^{\varepsilon\rho}\tilde{R}_{\alpha\varepsilon\kappa\rho} + T_{\kappa}{}^{\varepsilon\rho}\tilde{R}_{\alpha\varepsilon\beta\rho} + T_{\kappa\beta}{}^{\rho}\tilde{R}_{\alpha\rho}\} \tag{A2.16b}$$

$$\{\tilde{R}_{;\kappa} - 2\tilde{R}_{\varepsilon\kappa;}{}^{\varepsilon}\} = \{T^{\alpha\beta\kappa}\tilde{R}_{\alpha\beta\varepsilon\kappa} + 2T_{\kappa}{}^{\alpha\beta}\tilde{R}_{\alpha\beta}\}. \tag{A2.16c}$$

The cyclic identity and its contractions are

$$\{\tilde{R}_{\alpha\beta\mu\nu} + \tilde{R}_{\alpha\mu\nu\beta} + \tilde{R}_{\alpha\nu\beta\mu}\} = \{-T_{\beta\mu\alpha;\nu} - T_{\mu\nu\alpha;\beta} - T_{\nu\beta\alpha;\mu} + T_{\beta\mu}{}^{\kappa}T_{\kappa\nu\alpha}$$
$$+ T_{\mu\nu}{}^{\kappa}T_{\kappa\beta\alpha} + T_{\nu\beta}{}^{\kappa}T_{\kappa\mu\alpha}\} \tag{A2.17a}$$

$$\{\tilde{R}_{\alpha\beta} - \tilde{R}_{\beta\alpha}\} = \{-T_{\alpha}{}^{\kappa}{}_{\kappa;\beta} - T_{\kappa\beta}{}^{\kappa}{}_{;\alpha} - T_{\beta\alpha\kappa;}{}^{\kappa} + T_{\beta\alpha}{}^{\kappa}T_{\kappa}{}^{\varepsilon}{}_{\varepsilon}\}. \tag{A2.17b}$$

The Ricci identity is

$$[\tilde{\nabla}_{\kappa}, \tilde{\nabla}_{\varepsilon}]A_{\alpha\ldots\beta} = \{Y_{\kappa\varepsilon}A_{\alpha\ldots\beta} + T_{\kappa\varepsilon}{}^{\rho}A_{\alpha\ldots\beta;\rho} + A_{\rho\ldots\beta}\tilde{R}_{\alpha}{}^{\rho}{}_{\kappa\varepsilon} + \ldots + A_{\alpha\ldots\rho}\tilde{R}_{\beta}{}^{\rho}{}_{\kappa\varepsilon}\}. \tag{A2.18}$$

To obtain the usual identities for the torsionless curvature tensor one simply sets the torsion equal to zero in equations (A2.16)-(A2.18) and takes off the tildes.

For the primed (double primed) identities, simply replace the tildes in (A2.16)-(A2.18) with primes (double primes), put primes (double primes) above all the torsion tensors and substitute derivatives with primed (double primed) derivatives.

The various contorsion tensors and torsion tensors are related by

$$\acute{\acute{K}}_{\alpha\beta\kappa} = \{\acute{K}_{\alpha\beta\kappa} - K_{\alpha\beta\kappa}\} \tag{A2.19a}$$

$$\acute{\acute{T}}_{\alpha\beta\kappa} = \{\acute{T}_{\alpha\beta\kappa} - T_{\alpha\beta\kappa}\} \tag{A2.19b}$$

$$\acute{T}_{\alpha\beta\kappa} = \acute{K}_{\alpha\kappa\beta}. \tag{A2.19c}$$

Appendix 3. Toy torsion manifold

$$\tilde{R}_{\alpha\beta\mu\nu} \equiv 0 \tag{A3.1}$$

$$T_{\alpha\beta\kappa;\varepsilon} = 0 \tag{A3.2}$$

$$T_{\alpha\beta\kappa} = T_{[\alpha\beta\kappa]} \neq 0. \tag{A3.3}$$

Using the cyclic identity for torsion (2.15) and the restrictions above it follows immediately that

$$|T_{\alpha\beta\kappa} T_{\mu\nu}{}^{\kappa}|^2 = 2T^{\alpha\beta\kappa} T^{\mu\nu}{}_{\kappa} T_{\alpha\mu}{}^{\varepsilon} T_{\beta\nu\varepsilon}. \tag{A3.4}$$

Using the derivatives of the geodetic interval (A1.2) through (A1.7), replacing in all derivatives, ';' with ';' and using the Ricci identity for derivatives with torsion we obtain

$$[\sigma] = 0 \tag{A3.5}$$

$$[\sigma_{;\kappa}] = 0 \tag{A3.6}$$

$$[\sigma_{;\kappa\varepsilon}] = g_{\kappa\varepsilon} \tag{A3.7}$$

$$[\sigma_{;\alpha\beta\kappa}] = -\tfrac{1}{2}T_{\alpha\beta\kappa} = -K_{\alpha\beta\kappa} \tag{A3.8}$$

$$[\sigma_{;\alpha\beta\kappa\varepsilon}] = \{K_{\kappa\varepsilon}{}^{\rho}K_{\alpha\beta\rho} - \tfrac{1}{3}K_{\beta\varepsilon\rho}K_{\alpha\kappa}{}^{\rho} - \tfrac{1}{3}K_{\beta\kappa\rho}K_{\alpha\varepsilon}{}^{\rho}\} \tag{A3.9a}$$

$$[\sigma_{;\kappa\alpha}{}^{\alpha}{}_{\varepsilon}] = [\sigma_{;\kappa\varepsilon\alpha}{}^{\alpha}] = [\sigma_{;\alpha}{}^{\alpha}{}_{\kappa\varepsilon}] = -\tfrac{2}{3}K_{\kappa\alpha\beta}K_{\varepsilon}{}^{\alpha\beta} \tag{A3.9b}$$

$$|[\sigma_{;\alpha\beta\kappa\varepsilon}]|^2 = \tfrac{4}{3}|K_{\kappa\alpha\beta}K_{\varepsilon}{}^{\alpha\beta}|^2 \tag{A3.9c}$$

$$[\sigma_{;\varepsilon\alpha}{}^{\alpha}{}_{\kappa}{}^{\kappa}] = [\sigma_{;\alpha}{}^{\alpha}{}_{\varepsilon\kappa}{}^{\kappa}] = [\sigma_{;\alpha}{}^{\alpha}{}_{\kappa}{}^{\kappa}{}_{\varepsilon}] = 0 \tag{A3.10a}$$

$$\begin{aligned}[][\sigma_{;}{}^{\varepsilon\mu\nu}][\sigma_{;\varepsilon\alpha}{}^{\alpha}{}_{\mu\nu}] &= [\sigma_{;}{}^{\varepsilon\mu\nu}][\sigma_{;\varepsilon\mu\alpha}{}^{\alpha}{}_{\nu}] = [\sigma_{;}{}^{\varepsilon\mu\nu}][\sigma_{;\varepsilon\mu\nu\alpha}{}^{\alpha}] \\ &= -\tfrac{2}{3}|K_{\kappa\alpha\beta}K_{\varepsilon}{}^{\alpha\beta}|^2\end{aligned} \tag{A3.10b}$$

$$[\sigma_{;\kappa}{}^{\kappa}{}_{\varepsilon}{}^{\varepsilon}{}_{\alpha}{}^{\alpha}] = 0. \tag{A3.11}$$

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